ECE 102 - Systems and Signals (made by Don D. Le)

INTRODUCTION

Signals vs Systems.

1. **Signal:** Function that carries information ($x(t)$).
2. **System:** Transforms input $x(t)$ into output $y(t) = S(x(t))$.

Signal Transformations.

1. Amplitude scaling: $x(t) \to Ax(t)$: scale the graph vertically, if $A < 0$, the the graph is flipped across the x-axis
2. Time scaling: $x(at) \to$ compress if $a > 1$, expand if $0 < a < 1$
3. Time reversal: $x(-t)$: reflects the graph of a $x(t)$ across the y-axis
4. Time shifting: $x(t - T)$ (delay), $x(t + T)$ (advance)

General rule of graphing a shifted/scaled graph: $x(2(x - 1))$

1. Expand the parentheses: $x(2x - 2)$.
2. Next shift. For this case, shift by 2
3. Finally scale. Compress the graph by a times to the left if $0 < a < 1$, and expand to the right if $a > 1$. (see lecture 2)

Even/Odd Decomposition.

$$x(t) = \frac{x(t) + x(-t)}{2} + \frac{x(t) - x(-t)}{2} = x_e(t) + x_o(t)$$

Periodicity.

$$f(t + T) = f(t), \quad T > 0$$

Fundamental Period. A fundamental period is the shortest duration over which a periodic function or signal repeats its pattern exactly. For example,

- $\sin(t)$ can have many many periods $(2\pi, 4\pi, 6\pi)$ because $\sin(t + k2\pi) = \sin(t)$ but it only has one fundamental period that is 2π
- $x(t) = 1$ is periodic because $x(t + T) = x(t) = 1$ for any T. However, it has no fundamental period because there is no smallest positive value of T that satisfies this condition.

COMPLEX NUMBERS RECAP

Basics.

$$z = a + jb = re^{j\theta}, \quad r = |z| = \sqrt{a^2 + b^2}, \ \theta = \tan^{-1}\frac{b}{a}$$

Note that

$$\theta = \begin{cases} \tan^{-1}\frac{b}{a}, & \text{for quadrant I} \\ \pi + \tan^{-1}\frac{b}{a}, & \text{for quadrant II} \\ \pi + \tan^{-1}\frac{b}{a}, & \text{for quadrant III} \\ \tan^{-1}\frac{b}{a} + 2\pi, & \text{for quadrant IV} \end{cases}$$

Euler's Formula.

$$e^{j\theta} = \cos\theta + j\sin\theta$$

$$\cos\theta = \frac{e^{j\theta} + e^{-j\theta}}{2}, \quad \sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}$$

Properties.

1. $z' = a - jb = re^{-j\theta}$
2. $z_1 z_2 = r_1 r_2 e^{j(\theta_1 + \theta_2)}$
3. $\frac{z_1}{z_2} = \frac{r_1}{r_2} e^{j(\theta_1 - \theta_2)}$
4. $\Re\{z\} = \frac{z + z^*}{2}, \quad \Im\{z\} = \frac{z - z^*}{2j}$

One can transform a complex number $a + bi$ from rectangular to polar coordinates by

$$z = a + bi = \sqrt{a^2 + b^2}\left(\frac{a}{\sqrt{a^2 + b^2}} + \frac{b}{\sqrt{a^2 + b^2}} i\right)$$

Notice that if we let $\theta = \tan^{-1}(b/a)$, then

$$z = r(\cos\theta + i\sin\theta)$$

Phasors. Represent sinusoid $A\cos(\omega t + \phi)$ as

$$x(t) = \Re\{Ae^{j(\omega t + \phi)}\}$$

SYSTEMS

Some properties:

1. Linear: a system is linear if for some constants a and b,

$$S(ax_1 + bx_2) = aS(x_1) + bS(x_2)$$

2. Time-invariant: A system is time-invariant if $x(t - \tau)$ implies that $y(t - \tau) = S(x(t - \tau))$.

3. Causality: a system is causal if its output only depends on the past and present values of the input. In other words, $y(t) = S(x(t))$ depends only on $x(t)$, or $x(t - t_0)$ for some $t_0 > 0$ and not $x(t + t_0)$. For example,

$$y(t) = \underbrace{\int_{-\infty}^{t} x(\tau)d\tau}_{\text{causal bc it goes up to } t} + \underbrace{\cos(\omega t)}_{\text{causal bc it is not } x(t)}$$

Let's say if we have $\cos(\omega(t+1))$ instead of $\cos(\omega t)$, that's fine and the system is still causal because it does not depend on $x(t)$.

Fact: a system is not causal if it has value everywhere. That is, if it starts before 0, then it's not causal, like for example if $y(t)$ is odd or even, it means $y(-t) = y(t)$ or $-y(t)$, so the system is not causal.

4. Stability: A system is bounded-input, bounded-output (BIBO) stable if every bounded input leads to a bounded output:

$$|x(t)| < \infty \Rightarrow |y(t)| < \infty$$

For example, $y(t) = x(3t) + \cos(\omega t)$. If $x(3t) < B$, then $x(3t) + \cos(\omega t) < B + 1$ and therefore the system is BIBO stable.

5. Memory: a system has memory if its output depends on past or future values of the input. For example, an integrator signal:

$$y(t) = \int_{-\infty}^{\infty} x(\tau)d\tau$$

6. Memoryless: a system is memoryless if its output $y(t)$ depends only on the current input. For example, an AM radio signal:

$$y(t) = x(t)\cos(\omega_L t)$$

Impulse Response and LTI.

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t - \tau)d\tau = (x * h)(t)$$

If $x(t) = \delta(t)$, then $y(t) = h(t)$.

Invertibility. A system is called invertible if an input can always be exactly recovered from the output. That is, a system S is invertible if $\exists S^{inv}$ such that

$$x(t) = S^{inv}[S(x(t))] = S^{inv}[y(t)]$$

Suppose that there exists a linear, invertible system S with inverse S^{inv}, then

$$S^{inv} = S^{-1}(x)$$

is also linear.

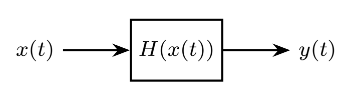
Linearity and TI recap.

Function	Linear?	TI?
$y(t) = \sqrt{x(t)}$	No	Yes
$y(t) = x(t) - z(t) \ (z(t) \neq 0)$	No	No
$y(t) = x(at)$	Yes	No
$y(t) = x(t - \tau)$	Yes	Yes
$y(t) = x(\tau - t)$	Yes	No

SYSTEM RESPONSES

What is a Response?

- The **response** of a system refers to the output signal $y(t)$ resulting from a given input $x(t)$.
- In LTI systems, the response fully characterizes how the system behaves for any input.
- It is often decomposed into parts to analyze effects of input and initial conditions.



Cascaded and Parallel Responses.

- The series cascade of any two time-invariant systems is also time-invariant
- The parallel cascade of any two time-invariant systems is also time-invariant

Types of Responses.

1. **Impulse Response:** The system's output when the input is $\delta(t)$.

$$h(t) = S[\delta(t)]$$

Represents how the system reacts to an instantaneous input.

2. **Step Response:** The system's output when the input is the unit step $u(t)$.

$$s(t) = h(t) * u(t)$$

Useful for analyzing transient and steady-state behavior.

IMPULSE RESPONSES

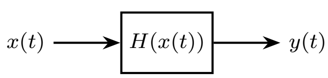
Definition. The impulse response is defined as $h(t) = H(x(t))$, where $x(t) = \delta(t)$.

Why important? It lets you predict the reponse (output) of the system to any input, assuming that this system is LTI.

How?

$$\begin{aligned} h(t) = x(t) * \delta(t) &= \int_{-\infty}^{\infty} x(\tau)\delta(\tau - t)d\tau \\ &= \int_{-\infty}^{\infty} x(t)\delta(\tau - t)d\tau \text{ (imuplse sambling property)} \\ &= x(t)\int_{-\infty}^{\infty} \delta(\tau - t)d\tau = x(t) \end{aligned}$$

If we know $h(t) = H(\delta(t))$, then given $x(t)$, we can find $y(t)$, assuming H is LTI.



$$\begin{aligned} y(t) = H(x(t)) &= H\left[\int_{-\infty}^{\infty} x(\tau)\delta(t - \tau)d\tau\right] \\ &= \int_{-\infty}^{\infty} x(\tau)H[\delta(t - \tau)]d\tau \text{ (linearity)} \\ &= \int_{-\infty}^{\infty} x(\tau)h(t - \tau)d\tau \text{ (time-invariance)} \\ &= x(t) * h(t) \end{aligned}$$

POWER AND ENERGY SIGNALS

Energy Signal.

$$E_x = \lim_{T\to\infty} \int_{-T}^{T} |x(t)|^2 dt$$

Power Signal.

$$P_x = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 dt$$

Power of Energy Signal?

- If $0 < E_x < \infty$, then $x(t)$ is energy signal
- If $0 < P_x < \infty$, then $x(t)$ is power signal
- If $E_x = C$, then $P_x = 0$

A signal can only be either power or energy signal, or neither, but can never be both.

DELTA FUNCTION

Definition. The Delta "function" is a distribution defined by

$$\delta(t) = \begin{cases} \infty & t = 0 \\ 0 & t \neq 0 \end{cases}, \quad \int_{-\infty}^{\infty} \delta(t)\,dt = 1$$

Sifting Property.

$$\int_{-\infty}^{\infty} x(t)\delta(t - t_0)dt = x(t_0)$$

$$\int_{-\infty}^{t} x(t)\delta(t - t_0)dt = x(t_0)u(t - t_0)$$

Impulse Sambling Property.

$$f(t)\delta(t) = f(0)\delta(t)$$

Basically, we only cares about f when $t = 0$, since it's 0 everywhere else due to the δ.

Key Identities.

1. $\delta(-t) = \delta(t)$
2. $\delta(at) = \frac{1}{|a|}\delta(t)$
3. $\frac{d}{dt}u(t) = \delta(t)$
4. $u(t) = \int_{-\infty}^{t} \delta(\tau)d\tau$
5. $\delta(t - a) * \delta(t - b) = \delta(t - (a + b))$
6. Delta can be expressed as an infinite sum

$$\delta(f(t)) = \sum_{i=-\infty}^{\infty} \frac{\delta(t - t_i)}{|f'(t_i)|}, \text{ where } f'(t_i) = 0$$

Examples.

$$x(t) = 2\delta(t + 1) + 3\delta(t - 2) \Rightarrow \int x(t)\,dt = 2 + 3 = 5$$

CONVOLUTION

Definition.

$$(f * g)(t) = \int_{-\infty}^{\infty} f(\tau)g(t - \tau)d\tau$$

Steps to Compute:

1. Flip one signal: $g(t) \to g(-\tau)$
2. Shift: $g(-\tau) \to g(t - \tau)$
3. Multiply: $f(\tau)g(t - \tau)$
4. Integrate over τ

Properties. Basic properties include commutative, associative, and distributive, and

1. Identity: $f * \delta = f$
2. Scaling: $f(at) * g(at) = \frac{1}{|a|}h(at)$, where $h(t) = f(t) * g(t)$

Common Results.

1. $\int_{-\infty}^{t} u(\tau)d\tau = u(t) * u(t) = t\,u(t) = r(t)$
2. $e^{-at}u(t) * u(t) = \frac{1-e^{-at}}{a}u(t)$
3. $\delta(t - t_0) * x(t) = x(t - t_0)$
4. If $x(t) * y(t) = z(t)$, then $x(t - a) * y(t - b) = z(t - (a + b))$
5. The integrator:

$$\int_{-\infty}^{t} x(\tau)d\tau = (x * u)(t)$$

STANDARD SIGNALS

Unit Step Function.

$$u(t) = \begin{cases} 1, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

Properties:

- $\frac{d}{dt}u(t) = \delta(t)$
- $\int_{-\infty}^{t} \delta(\tau)\,d\tau = u(t)$
- $u(t - t_0)$ shifts step to $t = t_0$

Ramp Function.

$$r(t) = t\,u(t)$$

Derivative: $\frac{d}{dt}r(t) = u(t)$, *Integral:* $\int r(t)dt = \frac{t^2}{2}u(t)$

Rectangular Pulse.

$$\text{rect}_\Delta(t) = \begin{cases} 1/\Delta, & |t| \leq \frac{\Delta}{2} \\ 0, & |t| > \frac{\Delta}{2} \end{cases}$$

Fact: the area of the rectangle is always 1.

Triangle Function.

$$\Delta\left(\frac{t}{T}\right) = \begin{cases} 1 - \frac{|t|}{T}, & |t| \leq T \\ 0, & |t| > T \end{cases}$$

Can be written as: $\Delta(t/T) = T\,\text{rect}(t/T) * \text{rect}(t/T)$

Exponential Signal.

$$x(t) = e^{at}u(t)$$

Stable if $\Re(a) < 0$; *energy decays over time.*

Sinusoidal Signal.

$$x(t) = A\cos(\omega_0 t + \phi)$$

Period: $T_0 = \frac{2\pi}{\omega_0}$, *Frequency:* $f = \frac{1}{T_0} = \frac{\omega_0}{2\pi}$.

Periodicity of sum of two signals. Given two signals $x(t)$ and $y(t)$, each with period T_x and T_y, the sum of these signals

$$z(t) = x(t) + y(t)$$

is periodic if and only if $r = T_y/T_x = \frac{k_1}{k_2}$ is rational. This also works for $x(t)y(t)$. The general period is then

$$T = k_2 T_y = k_1 T_x$$

EIGENFUNCTIONS OF LTI SYSTEMS

Eigenfunctions. An eigenfunction (or eigensignal) of an LTI system H is a function $y(t)$ such that

$$y(t) = H(x(t)) = \lambda x(t), \quad \lambda \in \mathbb{C},$$

here, we know that $H(x(t)) = x(t) * h(t)$.

Exponentials as eigenfunctions of LTI systems. If $x(t) = e^{st}$, then

$$y(t) = \int_{\mathbb{R}} h(\tau) e^{s(t-\tau)} d\tau = \left[\int_{\mathbb{R}} h(\tau) e^{s\tau} d\tau\right] e^{st} = H(s) e^{st}.$$

Observe that $H(s)$ is the eigenvalue, and e^{st} is the eigenfunction. In general, if $x(t)$ is a linear combination of exponential complex signals,

$$x(t) = \sum_k a_k e^{s_k t},$$

we get

$$y(t) = H(x(t)) = \sum_k H(s_k) a_k e^{s_k t}$$

Example. Show that $f(t) = \cos(\omega_0 t)$ is not an eigenfunction of an LTI system.

FOURIER SERIES

Def. Let $f(t)$ be a complex-valued periodic function with fundamental period T_0. Then $f(t)$ can be represented by its complex Fourier series:

$$f(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t},$$

where the coefficients are given by

$$c_k = \frac{1}{T_0} \int_{t_0}^{t_0+T_0} f(t)\, e^{-jk\omega_0 t}\, dt.$$

Here, T_0 is the fundamental period, $\omega_0 = \frac{2\pi}{T_0}$ is the fundamental angular frequency, and the choice of t_0 is arbitrary as long as the interval length is T_0.

Orthogonality of Complex Exponentials. Recall that the inner product of complex functions $u(t)$ and $v(t)$ is defined by

$$\langle u, v \rangle = \int_a^b u(t)\overline{v(t)}\, dt,$$

The functions $f_k = e^{jk\omega_0 t}$ form an orthogonal set over any interval of length T_0:

$$\langle f_k, f_m \rangle = \frac{1}{T_0} \int_{t_0}^{t_0+T_0} e^{jk\omega_0 t}\, e^{-jm\omega_0 t}\, dt = \begin{cases} 1, & k = m, \\ 0, & k \neq m. \end{cases}$$

Uniqueness of the Coefficients. Because of this orthogonality, each coefficient c_k is uniquely determined by $f(t)$.

Convergence (Informal). If $f(t)$ is periodic and satisfies standard Fourier conditions (piecewise continuous, finite number of maxima/minima and discontinuities within one period), then the Fourier series converges to

$$f(t) \quad \text{at points where } f \text{ is continuous,}$$

and to

$$\frac{f(t^-) + f(t^+)}{2}$$

at points of discontinuity (Gibbs phenomenon may occur).

Parseval's Theorem. For periodic signals,

$$P = \frac{1}{T_0} \int_{t_0}^{t_0+T_0} |f(t)|^2\, dt = \sum_{k=-\infty}^{\infty} |c_k|^2.$$

This shows how the energy of the signal is distributed among its frequency components.

Relation to Real Fourier Series. The complex exponential form is equivalent to the trigonometric representation involving $\cos(k\omega_0 t)$ and $\sin(k\omega_0 t)$. The complex form is often preferred because of its compact notation and algebraic convenience.

PROPERTIES OF FOURIER SERIES

Average of the signal. c_0 is the average of the signal. Note that for $k = 0$, we have that

$$c_0 = \frac{1}{T_0} \int_{t_0}^{t_0+T_0} f(t) dt.$$

Thus, c_0 is the time-averaged mean of the signal and correspond to a constant value. For this reason, it's sometimes called the "DC component." DC means Direct Current and refers to non-alternating (non-sinusoidal) currents.

Complex representation. In general, c_k maybe complex, so they can be expressed as

$$c_k = \Re(c_k) + j\Im(c_k) = |c_k| e^{j\angle c_k}.$$

Here, $|c_k|$ is called the amplitude spectrum, and $\angle c_k$ is called the phase spectrum.

- Amplitude spectrum: how much magnitude of $k\omega_0$ is in the signal
- Phase spectrum: Mixture of sines and cosines

Symmetry. We can apply Euler's formula to rewrite the fourier series coefficient and review symmetries:

$$\begin{aligned} c_k &= \frac{1}{T_0} \int_{t_0}^{t_0+T_0} f(t) e^{-j\frac{2\pi k t}{T_0}}\, dt \\ &= \frac{1}{T_0} \int_{t_0}^{t_0+T_0} f(t) \left[\cos\left(\frac{2\pi k t}{T_0}\right) - j \sin\left(\frac{2\pi k t}{T_0}\right)\right] dt \\ &= \frac{1}{T_0} \int_{t_0}^{t_0+T_0} f(t) \cos\left(\frac{2\pi k t}{T_0}\right) dt - \frac{j}{T_0} \int_{t_0}^{t_0+T_0} f(t) \sin\left(\frac{2\pi k t}{T_0}\right) dt \end{aligned}$$

Real Symmetry. If $f(t)$ is real, then

$$\Re(c_k) = \frac{1}{T_0} \int_{t_0}^{t_0+T_0} f(t) \cos\left(\frac{2\pi k}{T_0} t\right) dt$$
$$\Im(c_k) = -\frac{1}{T_0} \int_{t_0}^{t_0+T_0} f(t) \sin\left(\frac{2\pi k}{T_0} t\right) dt$$

For $f(t)$ real, and using the fact that $\cos(t)$ is even and $\sin(t)$ is odd, we have

$$\Re(c_k) = \Re(c_{-k}) \qquad \Im(c_k) = -\Im(c_{-k})$$
$$\overline{c_k} = c_{-k} \qquad |c_k| = |c_{-k}|$$
$$\angle c_k = -\angle \overline{c_k}$$

Real and Even/Odd Symmetry.

- If $f(t)$ is real and even $\quad \Rightarrow \quad c_k = c_{-k} \in \mathbb{R}$.
- If $f(t)$ is real and odd $\quad \Rightarrow \quad c_k = -c_{-k}, \quad c_k \in j\mathbb{R}$.

Complex Symmetry. If $f(t)$ is complex, let $f(t) = jg(t)$. Then,

$$\Re(c_k) = \frac{1}{T_0} \int_{t_0}^{t_0+T_0} g(t) \sin\left(\frac{2\pi k}{T_0} t\right) dt$$
$$\Im(c_k) = -\frac{1}{T_0} \int_{t_0}^{t_0+T_0} g(t) \cos\left(\frac{2\pi k}{T_0} t\right) dt$$

For $f(t)$ complex, and using the fact that $\cos(t)$ is even and $\sin(t)$ is odd, we have

$$\Re(c_k) = -\Re(c_{-k}) \qquad \Im(c_k) = \Im(c_{-k})$$
$$c_{-k} = -\overline{c_k} \qquad |c_{-k}| = |\overline{c_k}|$$

Dirichlet Conditions (informal). A periodic function $f(t)$ with period T_0 has a Fourier series representation if, over one period,

- $f(t)$ is absolutely integrable,
- $f(t)$ has a finite number of maxima and minima,
- $f(t)$ has a finite number of discontinuities.

These conditions guarantee the Fourier series converges to $f(t)$ almost everywhere.

Time Shift. If $f(t)$ has Fourier coefficients c_k, then shifting the signal in time by t_0 gives

$$f(t - t_0) \longleftrightarrow c_k e^{-jk\omega_0 t_0}.$$

A time shift introduces a linear phase term.

Time Reversal.

$$f(-t) \longleftrightarrow c_{-k}.$$

Linearity. For $f_1(t)$ and $f_2(t)$ with coefficients $c_k^{(1)}$ and $c_k^{(2)}$,

$$af_1(t) + bf_2(t) \longleftrightarrow ac_k^{(1)} + bc_k^{(2)}.$$

Differentiation in Time. Let $f'(t) = \frac{d}{dt} f(t)$. If $f(t) \longleftrightarrow c_k$, then

$$f'(t) \longleftrightarrow (jk\omega_0) c_k.$$

Integration in Time. If $g(t) = \int_{-\infty}^{t} f(\tau)\, d\tau$, then

$$g(t) \longleftrightarrow \frac{c_k}{jk\omega_0}, \qquad k \neq 0.$$

(For $k = 0$, integration introduces a constant that must be handled separately.)

Convolution Property for Periodic Signals. If $x(t)$ and $h(t)$ are periodic with Fourier coefficients $X[k]$ and $H[k]$, then their convolution over one period has coefficients

$$(x * h)(t) \longleftrightarrow X[k]\, H[k].$$

This mirrors the convolution property of Fourier transforms, but in a discrete-frequency domain.

Frequency Response from Fourier Series. For periodic inputs composed of harmonics,

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t},$$

passing through an LTI system with frequency response $H(j\omega)$ yields

$$y(t) = \sum_{k=-\infty}^{\infty} H(jk\omega_0) c_k e^{jk\omega_0 t}.$$

Each harmonic is scaled by the system's frequency response at that frequency.

FOURIER TRANSFORM

Motivation. The standard Fourier series that we know of only applies to periodic functions. However, this concept of writing a function as a series can be expanded over the entire real line, $-\infty < x < \infty$ if we let $T_0 \to \infty$.

Define the truncated Fourier transform to be

$$F_T(\omega) := \int_{-T/2}^{T/2} f(t) e^{-j\omega t} dt$$

We can relate this with the Fourier series by noticing that

$$c_k = \frac{1}{T} F_T(k\omega_0)$$

We can now reconstruct signal $f_T(t)$

$$f_T(t) = \sum_{k=-\infty}^{\infty} \frac{1}{T} F_T(k\omega_0) e^{jk\omega_0 t}$$

The Fourier transform is the limit of the truncated F.T.

$$F(\omega) = \lim_{T\to\infty} F_T(\omega)$$

Inverting the Fourier Transform. How do we go from $F(\omega)$ back to $f(t)$? We begin by writing the Fourier seriese

$$f(t) = \lim_{T\to\infty} f_T(t) = \lim_{T\to\infty} \sum_{k=-\infty}^{\infty} \frac{1}{T} F_T(k\omega_0) e^{jk\omega_0 t}$$

The Fourier Transform. The Fourier transform of a function $f(t)$ is defined as

$$\mathcal{F}\{f\}(\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt$$

The Inverse Fourier Transform. The inverse Fourier transform of of a transform $\mathcal{F}(\omega)$ is defined as

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathcal{F}(\omega) e^{j\omega t} dt$$

PROPERTIES OF FOURIER TRANSFORM

Linearity. For any signals $f(t)$ and $g(t)$, and constants $a, b \in \mathbb{C}$,

$$\mathcal{F}\{af(t) + bg(t)\} = aF(\omega) + bG(\omega).$$

Time Shifting. If $f(t) \longleftrightarrow F(\omega)$, then

$$f(t - t_0) \longleftrightarrow e^{-j\omega t_0} F(\omega).$$

Frequency Shifting.

$$e^{j\omega_0 t} f(t) \longleftrightarrow F(\omega - \omega_0).$$

Scaling. If $a \neq 0$,

$$f(at) \longleftrightarrow \frac{1}{|a|} F\left(\frac{\omega}{a}\right).$$

Differentiation in Time.

$$\frac{d}{dt} f(t) \longleftrightarrow j\omega F(\omega).$$

Differentiation in Frequency.

$$t\, f(t) \longleftrightarrow j \frac{dF(\omega)}{d\omega}.$$

Convolution Property. If $y(t) = (f * g)(t)$, then

$$Y(\omega) = F(\omega) G(\omega).$$

Multiplication Property.

$$f(t)g(t) \longleftrightarrow \frac{1}{2\pi} F(\omega) * G(\omega).$$

Energy of Aperiodic Signals (Parseval).

$$\int_{-\infty}^{\infty} |f(t)|^2\, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |F(\omega)|^2\, d\omega.$$

DISCRETE FOURIER TRANSFORM (DFT)

Motivation. Let $f(t)$ be a continuous signal. The Fourier Transform of the signal $f(t)$ is

$$F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt$$

We could regard each sample $f[k]$ as an impulse having area $f[k]$. Then, since the interand exists only at the sample points:

$$\begin{aligned} F(\omega) &= \int_{0}^{(N-1)T} f(t) e^{-j\omega t} dt \\ &= f[0]e^{-j0} + f[1]e^{-j\omega T} + \ldots f[N-1]e^{-j\omega(N-1)T} \\ &= \sum_{n=0}^{N-1} f[n] e^{-j\omega kT} \end{aligned}$$

The continous Fourier transform cold be evaluated over a finite interval (usually over the fundamental period T_0) rather than from $-\infty$ to $+\infty$ if the waveform is periodic. Similarly, since there can only be many finite data points input, the DFT treats the data as if they were periodic. Meaning, the values of $f(N)$ to $f(2N-1)$ is the same as $f(0)$ to $f(N-1)$. Let

$$\omega = k\frac{2\pi}{NT}, \quad k = 0, 1, \ldots, N-1,$$

we arrive at

$$F[k] = \sum_{n=0}^{N-1} f[n] e^{-j\frac{2\pi}{N}kn}$$

Definition. Given a finite-length discrete sequence $x[n]$ of length N, the DFT is defined by

$$X[k] = \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi}{N}kn}, \qquad k = 0, 1, \ldots, N-1.$$

The inverse DFT is

$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k] e^{j\frac{2\pi}{N}kn}.$$

Interpretation. The DFT converts a finite discrete-time signal into a discrete frequency representation. The output $X[k]$ is periodic with period N.

Orthogonality of DFT Basis. The complex vectors $e^{-j\frac{2\pi}{N}kn}$ are orthogonal over $n = 0, 1, \ldots, N-1$:

$$\sum_{n=0}^{N-1} e^{-j\frac{2\pi}{N}(k-m)n} = \begin{cases} N, & k = m, \\ 0, & k \neq m. \end{cases}$$

FAST FOURIER TRANSFORM (FFT)

1 Why FFT Exists (Computational View)

The DFT is a linear transform:

$$X[k] = \sum_{n=0}^{N-1} x[n] W_N^{kn}, \qquad W_N = e^{-j\frac{2\pi}{N}}.$$

Computing this directly requires:

- N outputs,
- each output uses N complex multiplications,

for a total cost of

$$\boxed{O(N^2)}.$$

This becomes impractical even for moderate N (e.g. images, audio, real-time systems).

2 Roots of Unity and Symmetry

Define the primitive N-th root of unity:

$$W_N = e^{-j\frac{2\pi}{N}}.$$

Key identities:

$$W_N^{k+N} = W_N^k, \qquad W_N^{2k} = W_{N/2}^k, \qquad W_N^{k+N/2} = -W_N^k.$$

These identities allow reuse of computations across frequency bins.

3 Even–Odd Decomposition (Core FFT Idea)

Assume N is even. Split the signal into even and odd indexed samples:

$$x_e[n] = x[2n], \qquad x_o[n] = x[2n+1], \quad n = 0, \dots, \frac{N}{2} - 1.$$

Start from the DFT:

$$X[k] = \sum_{n=0}^{N-1} x[n] W_N^{kn}.$$

Separate even and odd terms:

$$\begin{aligned} X[k] &= \sum_{n=0}^{N/2-1} x[2n] W_N^{k(2n)} + \sum_{n=0}^{N/2-1} x[2n+1] W_N^{k(2n+1)} \\ &= \sum_{n=0}^{N/2-1} x_e[n] W_{N/2}^{kn} + W_N^k \sum_{n=0}^{N/2-1} x_o[n] W_{N/2}^{kn}. \end{aligned}$$

Define:

$$E[k] = \sum_{n=0}^{N/2-1} x_e[n] W_{N/2}^{kn}, \qquad O[k] = \sum_{n=0}^{N/2-1} x_o[n] W_{N/2}^{kn}.$$

Then:

$$\boxed{X[k] = E[k] + W_N^k O[k]}$$

Using periodicity:

$$\boxed{X[k+N/2] = E[k] - W_N^k O[k]}$$

This computes two outputs from the same sub-results.

4 Recursive Structure

Each N-point DFT reduces to:

- Two $(N/2)$-point DFTs,
- N additional complex multiplications (twiddle factors).

Recursively applying this until $N = 1$ gives:

$$T(N) = 2T(N/2) + O(N).$$

By the Master Theorem:

$$\boxed{T(N) = O(N \log_2 N)}.$$

5 Butterfly Operation

The basic FFT computation unit is the *butterfly*. Given inputs (a, b):

$$\begin{aligned} y_0 &= a + Wb, \\ y_1 &= a - Wb, \end{aligned}$$

where W is a twiddle factor.

Properties.

- In-place computation.
- Same operation used at every stage.
- Hardware-friendly.

Every FFT implementation is essentially a large collection of butterflies.

6 FFT Pseudocode (Radix-2, Recursive)

Assume $N = 2^m$.

```
FFT(x):
    N = length(x)
    if N == 1:
        return x

    x_even = x[0], x[2], x[4], ...
    x_odd  = x[1], x[3], x[5], ...

    E = FFT(x_even)
    O = FFT(x_odd)

    for k = 0 to N/2 - 1:
        W = exp(-j*2*pi*k/N)
        X[k]       = E[k] + W*O[k]
        X[k+N/2]   = E[k] - W*O[k]

    return X
```

This version is simple conceptually but uses extra memory.

7 Iterative FFT and Bit Reversal

Practical FFTs are iterative and in-place.

Bit-reversal permutation. Indices are reordered so that recursion becomes iteration.

Example for $N = 8$:

Index	Bit-reversed
000	000
001	100
010	010
011	110
100	001
101	101
110	011
111	111

After bit-reversal, butterflies are applied stage by stage.

8 Decimation-in-Time vs Decimation-in-Frequency

DIT FFT.

- Splits input into even/odd samples.
- Bit-reversal occurs on input.

DIF FFT.

- Splits output into even/odd frequencies.
- Bit-reversal occurs on output.

Both compute the same DFT with identical complexity.

9 Numerical and Practical Considerations

Complexity.

$$\text{Multiplications} \approx \frac{N}{2} \log_2 N, \qquad \text{Additions} \approx N \log_2 N.$$

Cache and memory. FFT performance is often limited by memory access, not arithmetic.

Real-valued signals. If $x[n]$ is real:

$$X[N-k] = X^*[k],$$

so only half the spectrum is independent.

Zero-padding.

- Does not add information.
- Interpolates the frequency grid.
- Useful for visualization and peak localization.

10 Conceptual Summary

- The FFT is not a transform, but an algorithm.
- It exploits symmetry of roots of unity.
- Even–odd decomposition is the key mathematical step.
- Butterflies are the atomic operations.
- $O(N \log N)$ makes modern DSP possible.

FILTERS

Big picture. Fourier transform converts a signal into a sum of complex exponentials. An LTI system responds to each complex exponential in a very simple way, so:

- Analyze the system in frequency.
- Multiply spectra instead of convolving in time.
- See how each frequency is scaled and phase shifted.

Frequency response definition. Let $h(t)$ be the impulse response. The frequency response is

$$H(j\omega) = \mathcal{F}\{h(t)\} = \int_{-\infty}^{\infty} h(t) e^{-j\omega t} dt.$$

Eigenfunction property. If the input is a complex sinusoid,

$$x(t) = e^{j\omega_0 t},$$

then the output of any LTI system is

$$y(t) = H(j\omega_0) e^{j\omega_0 t}.$$

Interpretation:

- The frequency stays the same (ω_0).
- Amplitude is scaled by $|H(j\omega_0)|$.
- Phase is shifted by $\angle H(j\omega_0)$.

Input/output in frequency. For general input $x(t)$ with spectrum $X(j\omega)$:

$$Y(j\omega) = H(j\omega)X(j\omega).$$

In terms of magnitude and phase,

$$|Y(j\omega)| = |H(j\omega)|\,|X(j\omega)|, \qquad \angle Y(j\omega) = \angle H(j\omega) + \angle X(j\omega).$$

Example: RC low-pass circuit. For an RC low-pass filter (with suitable choice of output node):

$$H(j\omega) = \frac{1}{1 + j\omega RC}.$$

Study three regimes:

- **Low frequency** $\omega \ll \frac{1}{RC}$: $H(j\omega) \approx 1$ Output $\approx$ input (low frequencies pass).
- **Cutoff region** $\omega \approx \frac{1}{RC}$: $|H| = \frac{1}{\sqrt{2}}$ at $\omega = 1/RC$ (about -3 dB).
- **High frequency** $\omega \gg \frac{1}{RC}$: $|H| \to 0$ High-frequency components are heavily attenuated.

IDEAL FILTERS

Ideal low-pass filter (LPF).

$$H_{\text{LP}}(j\omega) = \begin{cases} 1, & |\omega| \le \omega_c, \\ 0, & |\omega| > \omega_c. \end{cases}$$

The impulse response is the inverse Fourier transform:

$$h_{\text{LP}}(t) = \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} e^{j\omega t}d\omega = \frac{\sin(\omega_c t)}{\pi t}.$$

Properties:

- $h_{\text{LP}}(t)$ decays like $1/t$ but never becomes exactly zero.
- It is non-causal (extends to negative time).
- Infinite support: cannot be implemented exactly, but can be approximated.

Ideal high-pass filter (HPF).

$$H_{\text{HP}}(j\omega) = \begin{cases} 0, & |\omega| < \omega_c, \\ 1, & |\omega| \ge \omega_c. \end{cases}$$

Ideal band-pass filter. Passes only frequencies in an interval around some center frequency:

$$H_{\text{BP}}(j\omega) = \begin{cases} 1, & \omega_1 \le |\omega| \le \omega_2, \\ 0, & \text{otherwise}. \end{cases}$$

Delta through a filter. Since $\delta(t) \longleftrightarrow 1$,

$$\delta(t) \xrightarrow{\text{system}} h(t),$$

so passing a delta through any system gives its impulse response. Passing it through an ideal LPF gives $h_{\text{LP}}(t)$ above.

REALIZABLE FILTERS: GENERAL CONCEPTS

Practical filters relax ideal constraints. Real filters trade:

- Sharper transitions **vs.** shorter impulse responses.
- Flat passbands **vs.** simpler implementations.
- Linear phase **vs.** minimum delay.

Key specifications.

- Passband edge ω_p.
- Stopband edge ω_s.
- Passband ripple.
- Stopband attenuation.
- Transition bandwidth $\omega_s - \omega_p$.

FILTER FAMILIES

11 Gaussian Filter

Frequency response.

$$H_{\text{G}}(j\omega) = e^{-\frac{\omega^2}{2\sigma^2}}.$$

Impulse response. The inverse Fourier transform is also Gaussian:

$$h_{\text{G}}(t) = \frac{\sigma}{\sqrt{2\pi}}e^{-\frac{\sigma^2 t^2}{2}}.$$

Key properties.

- Perfect smoothness in both time and frequency.
- No ringing (no Gibbs phenomenon).
- Minimizes joint time–frequency spread (optimal uncertainty).

Downside.

- No sharp cutoff.
- Poor selectivity compared to ideal LPF.

Gaussian filters are common in denoising and image processing, where smoothness is preferred over sharp frequency separation.

12 Butterworth Filter

Goal. Maximally flat magnitude response in the passband.
Low-pass magnitude response.

$$|H(j\omega)|^2 = \frac{1}{1 + \left(\frac{\omega}{\omega_c}\right)^{2n}},$$

where n is the filter order.
Properties.

- No ripple in passband or stopband.
- Higher n gives sharper transition.
- Poles lie on a circle in the left half-plane.

Trade-off.

- Smooth magnitude.
- Nonlinear phase.

Butterworth filters are often the default choice when a smooth response is needed.

13 Chebyshev Filters

Type I (Chebyshev I). Allows ripple in passband for sharper cutoff.

$$|H(j\omega)|^2 = \frac{1}{1 + \epsilon^2 T_n^2\left(\frac{\omega}{\omega_c}\right)},$$

where $T_n(\cdot)$ is a Chebyshev polynomial.
Properties.

- Sharper transition than Butterworth for same order.
- Equiripple passband.
- Monotone stopband.

Type II (Inverse Chebyshev).

- Ripple in stopband.
- Flat passband.

14 Comparision

Filter	Ripple	Transition	Phase
Ideal	None	Infinite sharp	Noncausal
Gaussian	None	Very smooth	Linear
Butterworth	None	Moderate	Nonlinear
Chebyshev I	Passband	Sharp	Nonlinear
Chebyshev II	Stopband	Sharp	Nonlinear
Elliptic	Both	Sharpest	Nonlinear

LINEAR PHASE AND GROUP DELAY

Phase distortion. Even if $|H(j\omega)|$ is ideal, nonlinear phase distorts waveforms.
Group delay.

$$\tau_g(\omega) = -\frac{d}{d\omega}\angle H(j\omega).$$

Linear phase condition.

$$\angle H(j\omega) = -\omega\tau_0 \quad \Rightarrow \quad \tau_g(\omega) = \tau_0 \text{ (constant)}.$$

Linear phase filters delay all frequency components equally and preserve waveform shape.

NOISE AND FILTERING

Suppose a channel adds noise:

$$y(t) = x(t) + n(t).$$

In frequency:

$$Y(j\omega) = X(j\omega) + N(j\omega).$$

If the noise is wideband (spreads across all frequencies), but the signal occupies only a narrow band, you can design a filter $H(j\omega)$ that:

- Keeps the band where $X(j\omega)$ is concentrated.
- Attenuates frequencies where only $N(j\omega)$ is present.

This is the basic idea of using filters for denoising.

However, ideal filters are not realizable; real filters have transition bands and non-flat passbands, causing distortion and incomplete noise removal.

NOISE, SNR, AND FILTER DESIGN

Additive noise model.

$$y(t) = x(t) + n(t).$$

After filtering.

$$Y(j\omega) = H(j\omega)X(j\omega) + H(j\omega)N(j\omega).$$

SNR improvement principle.

- Keep frequencies where $X(j\omega)$ is strong.
- Suppress frequencies dominated by $N(j\omega)$.

Fundamental limit. Aggressive filtering improves noise suppression but:

- Distorts signal near band edges.
- Introduces ringing or delay.

UNIFORM SAMPLING WITH AN IMPULSE TRAIN

F.T. of periodic signals. For a periodic $f(t)$ with period T_0, the Fourier transform is a line spectrum:

$$F(j\omega) = 2\pi \sum_{k=-\infty}^{\infty} c_k \delta(\omega - k\omega_0), \qquad \omega_0 = \frac{2\pi}{T_0},$$

where c_k are the complex Fourier series coefficients.

Key dual viewpoint:

- Periodic in time $\Rightarrow$ discrete in frequency (lines).
- Discrete in time $\Rightarrow$ periodic in frequency (repeated copies).

Impulse train and sampling rate.

$$\delta_T(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT), \qquad \omega_s = \frac{2\pi}{T}, \qquad f_s = \frac{1}{T}.$$

Note: $\delta(t)$ has units 1/s, so $\delta_T(t)$ also has 1/s.

Ideal (point) sampling as multiplication. Sampling a continuous-time signal $f(t)$ every T seconds is modeled by

$$f_s(t) = f(t)\,\delta_T(t) = \sum_{k=-\infty}^{\infty} f(kT)\,\delta(t - kT).$$

This is a distribution identity: multiplying by $\delta(t - kT)$ extracts the value at $t = kT$.

Samples as a sequence. Define the discrete-time sequence

$$x[k] \triangleq f(kT).$$

Then

$$f_s(t) = \sum_{k=-\infty}^{\infty} x[k]\;\delta(t - kT).$$

So ideal sampling converts a function $f(t)$ into a weighted impulse train whose weights are the sample values.

FOURIER TRANSFORM OF AN IMPULSE TRAIN

CTFT convention. Use

$$F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}\,dt, \qquad f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{j\omega t}\,d\omega.$$

FT of a shifted impulse and of the train.

$$\mathcal{F}\{\delta(t - t_0)\} = e^{-j\omega t_0}.$$

Hence

$$\mathcal{F}\{\delta_T(t)\} = \sum_{k=-\infty}^{\infty} e^{-j\omega kT}.$$

This object is periodic in ω with period $\omega_s = \frac{2\pi}{T}$ and equals (in the distribution sense)

$$\sum_{k=-\infty}^{\infty} e^{-j\omega kT} = \frac{2\pi}{T}\sum_{m=-\infty}^{\infty} \delta(\omega - m\omega_s),$$

so

$$\boxed{\mathcal{F}\{\delta_T(t)\} = \frac{2\pi}{T}\sum_{m=-\infty}^{\infty} \delta(\omega - m\omega_s)}.$$

Interpretation. Time-domain discreteness $\Rightarrow$ frequency-domain periodicity. The spacing between spectral lines is ω_s.

Spectrum of the sampled signal (replication). Using multiplication–convolution duality:

$$\mathcal{F}\{f(t)\delta_T(t)\} = \frac{1}{2\pi}\big(F * \mathcal{F}\{\delta_T\}\big)(\omega).$$

Substitute the impulse-train spectrum:

$$F_s(\omega) = \frac{1}{2\pi}\left(F * \frac{2\pi}{T}\sum_m \delta(\omega - m\omega_s)\right)(\omega) = \frac{1}{T}\sum_{m=-\infty}^{\infty} F(\omega - m\omega_s).$$

$$\boxed{F_s(\omega) = \frac{1}{T}\sum_{m=-\infty}^{\infty} F(\omega - m\omega_s)}$$

Sampling replicates $F(\omega)$ every ω_s and scales by $1/T$.

Why the $1/T$ factor matters. If you later select the baseband copy with an ideal LPF, you must undo this scaling by multiplying by T in the passband (see reconstruction section).

PERIODICITY / DISCRETENESS DUALITIES

Continuous-time periodic $\Rightarrow$ discrete spectrum. If $f(t)$ is periodic with fundamental period T_0, it has Fourier series

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn\omega_0 t}, \qquad \omega_0 = \frac{2\pi}{T_0},$$

and its CTFT is a line spectrum:

$$F(\omega) = 2\pi \sum_{n=-\infty}^{\infty} c_n\,\delta(\omega - n\omega_0).$$

Impulse train in time $\Rightarrow$ periodic in frequency. If $f(t) = \sum_k x[k]\delta(t - kT)$ (discrete in time), then $F(\omega)$ is $2\pi/T$ times a periodic function with period ω_s:

$$F(\omega) = \sum_k x[k]e^{-j\omega kT} \quad \Rightarrow \quad F(\omega + \omega_s) = F(\omega).$$

(Indeed $e^{-j(\omega+\omega_s)kT} = e^{-j\omega kT}e^{-j2\pi k} = e^{-j\omega kT}$.)

Sampling is the combination. Sampling turns a general $f(t)$ into an impulse train; therefore its spectrum becomes periodic, with the period set by ω_s.

NYQUIST, ALIASING, AND THE ALIAS FORMULA

Bandlimited assumption. Assume

$$F(\omega) = 0 \quad \text{for} \quad |\omega| > \omega_{\max}.$$

Nyquist condition (angular-frequency form). Replicas of $F(\omega)$ are centered at $m\omega_s$. No overlap if

$$\boxed{\omega_s > 2\omega_{\max}}.$$

Equivalently (Hz form) if $\Omega_{\max} = \omega_{\max}/(2\pi)$ and $f_s = \omega_s/(2\pi) = 1/T$:

$$\boxed{f_s > 2\Omega_{\max}}.$$

What aliasing *is* in frequency. When $\omega_s \le 2\omega_{\max}$, the shifted copies overlap:

$$F_s(\omega) = \frac{1}{T}\sum_m F(\omega - m\omega_s)$$

contains sums of different parts of $F(\omega)$ in the same frequency region. Once summed, they cannot be separated by any LTI filter.

Aliasing identity for sinusoids (time-domain intuition). Let

$$f(t) = \cos(\omega_0 t + \phi).$$

Samples:

$$f(kT) = \cos(\omega_0 kT + \phi).$$

For any integer m,

$$\cos\big((\omega_0 + m\omega_s)kT + \phi\big) = \cos(\omega_0 kT + m(2\pi)k + \phi) = \cos(\omega_0 kT + \phi),$$

so ω_0 and $\omega_0 + m\omega_s$ produce identical samples. Therefore, the sampled data cannot distinguish frequencies that differ by integer multiples of ω_s.

Principal alias frequency (mapping to baseband). A continuous frequency ω_0 will be observed after sampling as an aliased frequency

$$\omega_a = \omega_0 - m\omega_s$$

for some integer m that places ω_a in the chosen baseband interval. Common choices:

$$\omega_a \in \left[-\frac{\omega_s}{2}, \frac{\omega_s}{2}\right] \quad \text{or} \quad \omega_a \in [-\omega_{\max}, \omega_{\max}] \text{ (if Nyquist satisfied)}.$$

This is the formal statement behind the "different sinusoids give same samples" idea.

Anti-aliasing filter (practical note). Real systems prefilter with an analog LPF before sampling so that the input is effectively bandlimited:

$$|H_{\text{AA}}(j\omega)| \approx 1 \text{ for } |\omega| \le \omega_B, \qquad |H_{\text{AA}}(j\omega)| \approx 0 \text{ for } |\omega| > \omega_B,$$

and then enforce $\omega_s > 2\omega_B$.

RECONSTRUCTION (IDEAL) AND WHY IT WORKS

Goal. Given samples $x[k] = f(kT)$, recover $f(t)$ assuming no aliasing.

Sampled spectrum contains repeated baseband copies. With Nyquist satisfied, within $|\omega| \le \omega_{\max}$ we have

$$F_s(\omega) = \frac{1}{T}F(\omega) \quad \text{(because only the } m=0 \text{ term contributes in baseband).}$$

Ideal reconstruction LPF. Choose

$$H_{\text{rec}}(j\omega) = \begin{cases} T, & |\omega| \le \omega_{\max}, \\ 0, & \text{otherwise.} \end{cases}$$

Then in baseband:

$$Y(\omega) = H_{\text{rec}}(j\omega)F_s(\omega) = T \cdot \frac{1}{T}F(\omega) = F(\omega),$$

and outside baseband $Y(\omega) = 0$, consistent with $F(\omega)$ for a bandlimited signal.

Time-domain impulse response of the ideal LPF. Since $H_{\text{rec}}(j\omega)$ is a rectangle in frequency, its impulse response is a sinc. Compute:

$$h_{\text{rec}}(t) = \frac{1}{2\pi}\int_{-\omega_{\max}}^{\omega_{\max}} T\, e^{j\omega t}\, d\omega = \frac{T}{2\pi} \cdot \frac{2\sin(\omega_{\max} t)}{t} = \frac{T\omega_{\max}}{\pi} \cdot \frac{\sin(\omega_{\max} t)}{\omega_{\max} t}.$$

So

$$\boxed{h_{\text{rec}}(t) = \frac{T\omega_{\max}}{\pi}\,\text{sinc}_{\omega_{\max}}(t), \quad \text{sinc}_{\omega_{\max}}(t) \triangleq \frac{\sin(\omega_{\max} t)}{\omega_{\max} t}.}$$

Reconstruction as convolution.

$$\begin{aligned} f(t) = y(t) &= (f_s * h_{\text{rec}})(t) \\ &= \left(\sum_k x[k]\delta(t-kT)\right) * h_{\text{rec}}(t) \\ &= \sum_{k=-\infty}^{\infty} x[k]\; h_{\text{rec}}(t-kT). \end{aligned}$$

This is the general interpolation form: a sum of shifted kernels.

Whittaker–Shannon interpolation (Nyquist-rate form). If we choose the ideal cutoff at the Nyquist frequency, $\omega_{\max} = \omega_s/2 = \pi/T$, then

$$h_{\text{rec}}(t) = \frac{T(\pi/T)}{\pi} \cdot \frac{\sin\left(\frac{\pi}{T}t\right)}{\frac{\pi}{T}t} = \frac{\sin(\pi t/T)}{\pi t/T} = \text{sinc}\left(\frac{t}{T}\right),$$

where $\text{sinc}(u) = \frac{\sin(\pi u)}{\pi u}$. Therefore

$$\boxed{f(t) = \sum_{k=-\infty}^{\infty} f(kT)\,\text{sinc}\left(\frac{t-kT}{T}\right)}$$

when $f(t)$ is bandlimited to $|\omega| \le \pi/T$.

Key properties of sinc interpolation (useful facts).

- Interpolation: $\text{sinc}(n) = 0$ for nonzero integers n, and $\text{sinc}(0) = 1$. Hence $f(kT)$ is matched exactly at sample times.
- Basis view: $\{\text{sinc}((t-kT)/T)\}_{k\in\mathbb{Z}}$ form an interpolation basis for bandlimited signals.

ZERO-ORDER HOLD VS. IDEAL RECONSTRUCTION (PRACTICAL SIDENOTE)

Real DACs do not output impulses. A common model is a *zero-order hold* (ZOH): each sample value is held constant for one period. This corresponds to convolving the impulse train with a rectangular pulse:

$$p_{\text{ZOH}}(t) = u(t) - u(t-T), \qquad x_{\text{ZOH}}(t) = f_s(t) * p_{\text{ZOH}}(t).$$

Its frequency response is

$$P_{\text{ZOH}}(j\omega) = \int_0^T e^{-j\omega t}\, dt = Te^{-j\omega T/2}\,\text{sinc}\left(\frac{\omega T}{2\pi}\right) \quad \text{(up to your sinc convention).}$$

So ZOH introduces amplitude droop (a sinc-shaped magnitude) and a linear phase term. Ideal reconstruction would additionally equalize this droop if perfect recovery is desired.

COMMON PITFALLS / QUICK CHECKS

- **Be consistent about sinc.** Some texts use $\text{sinc}(x) = \frac{\sin x}{x}$, others use $\frac{\sin(\pi x)}{\pi x}$. Track which one you use; it changes constants in $h_{\text{rec}}(t)$.
- **Nyquist is strict inequality in theory.** $\omega_s = 2\omega_{\max}$ is a boundary case; any non-ideal filter transition band needs margin.
- **The scaling $1/T$ in $F_s(\omega)$ is not optional.** If you drop it, your reconstruction gain will be wrong by a factor of T.

THE LAPLACE TRANSFORM

15 Introduction

Limitation of Fourier. The Fourier transform requires

$$\int_{-\infty}^{\infty} |f(t)|\, dt < \infty \quad \text{(absolute integrability)}$$

or weaker energy conditions. Many physically important signals violate this:

$$e^{at}u(t), \quad a > 0, \qquad t^n u(t), \qquad \text{step responses of unstable systems.}$$

Key idea. Instead of analyzing signals purely along the imaginary axis ($s = j\omega$), extend frequency into the complex plane:

$$s = \sigma + j\omega.$$

The real part σ exponentially weights time:

$$e^{-st} = e^{-\sigma t}e^{-j\omega t}.$$

- $\sigma > 0$: suppresses growth as $t \to +\infty$.
- $\sigma < 0$: suppresses growth as $t \to -\infty$.
- $\sigma = 0$: recovers Fourier analysis.

Laplace is a *completion* of Fourier analysis.

16 Bilateral Laplace Transform

Definition.

$$F(s) = \mathcal{L}\{f(t)\} = \int_{-\infty}^{\infty} f(t)e^{-st}dt, \qquad s = \sigma + j\omega.$$

Interpretation.

- $F(s)$ is a complex-valued surface over the s-plane.
- Each vertical line $\Re(s) = \sigma$ corresponds to a Fourier transform of $f(t)e^{-\sigma t}$.

Fourier as a slice. If the line $\sigma = 0$ lies inside the ROC:

$$F(j\omega) = \mathcal{F}\{f(t)\}.$$

17 Region of Convergence (ROC)

Definition. The ROC is the set of all s such that

$$\int_{-\infty}^{\infty} |f(t)e^{-st}|dt < \infty.$$

Why ROC matters.

- The same algebraic expression $F(s)$ can correspond to *different* time-domain signals with different ROCs.
- Poles alone do not uniquely define a signal; poles + ROC do.

Example 1: right-sided exponential.

$$f(t) = e^{at}u(t).$$

$$F(s) = \int_0^{\infty} e^{(a-s)t}dt = \frac{1}{s-a}, \qquad \Re(s) > a.$$

Example 2: left-sided exponential.

$$f(t) = -e^{at}u(-t).$$

$$F(s) = \int_{-\infty}^{0} (-e^{at})e^{-st}dt = \frac{1}{s-a}, \qquad \Re(s) < a.$$

Same formula, different signals.

$$\boxed{F(s) = \frac{1}{s-a}}$$

but ROC determines causality.

General ROC shapes.

- Right-sided (causal): $\Re(s) > \sigma_0$.
- Left-sided: $\Re(s) < \sigma_0$.
- Two-sided: $\sigma_1 < \Re(s) < \sigma_2$.

18 Unilateral (One-sided) Laplace Transform

Definition. For causal signals:

$$\mathcal{L}\{f(t)\} = \int_0^{\infty} f(t)e^{-st}dt.$$

Why engineers prefer it.

- Automatically enforces causality.
- Initial conditions appear explicitly.
- Ideal for ODEs and physical systems.

Implicit assumption. $f(t) = 0$ for $t < 0$. The ROC is always a right half-plane.

19 Properties

Time shift.

$$\mathcal{L}\{f(t-t_0)u(t-t_0)\} = e^{-st_0}F(s).$$

Delay in time $\leftrightarrow$ exponential factor in s.
Frequency (s-domain) shift.

$$\mathcal{L}\{e^{at}f(t)\} = F(s-a).$$

Multiplying by e^{at} shifts poles by $+a$.
Differentiation in time.

$$\mathcal{L}\{f'(t)\} = sF(s) - f(0^-).$$

Derivatives introduce both multiplication by s and initial conditions.
Integration in time.

$$\mathcal{L}\left\{\int_0^t f(\tau)d\tau\right\} = \frac{1}{s}F(s).$$

Time-domain convolution.

$$\mathcal{L}\{f(t) * g(t)\} = F(s)G(s).$$

This makes Laplace ideal for LTI systems.

20 Laplace and LTI Systems

Impulse response view. For an LTI system with impulse response $h(t)$:

$$H(s) = \mathcal{L}\{h(t)\}$$

is the transfer function.
Input-output relation.

$$Y(s) = H(s)X(s).$$

Why Laplace simplifies everything.

$$\text{Differential equations} \longrightarrow \text{algebraic equations}$$

ODEs AND LAPLACE TRANSFORM

21 Introduction

Consider

$$y'' + a_1y' + a_0y = x(t).$$

Laplace transform.

$$(s^2Y - sy(0) - y'(0)) + a_1(sY - y(0)) + a_0Y = X(s).$$

Solve for $Y(s)$.

$$Y(s) = \frac{X(s)}{s^2 + a_1s + a_0} + \frac{sy(0) + y'(0) + a_1y(0)}{s^2 + a_1s + a_0}.$$

Interpretation.

- First term: forced response (input-driven).
- Second term: natural response (initial conditions).

The denominator polynomial defines system dynamics.

22 Simple Poles

Rational structure. Physical LTI systems produce

$$F(s) = \frac{b(s)}{a(s)},$$

with $\deg b < \deg a$ for causal systems. If

$$a(s) = \prod_{k=1}^{n}(s - \lambda_k), \quad \lambda_k \text{ distinct},$$

then

$$F(s) = \sum_{k=1}^{n} \frac{r_k}{s - \lambda_k}.$$

Residue formula.

$$r_k = \lim_{s \to \lambda_k}(s - \lambda_k)F(s).$$

Inverse Laplace.

$$\mathcal{L}^{-1}\left\{\frac{1}{s-\lambda}\right\} = e^{\lambda t}u(t).$$

Each pole produces exactly one exponential mode.

23 Repeated Poles

If

$$F(s) = \frac{r_1}{s-\lambda} + \frac{r_2}{(s-\lambda)^2} + \cdots + \frac{r_m}{(s-\lambda)^m},$$

then

$$\mathcal{L}^{-1}\left\{\frac{1}{(s-\lambda)^k}\right\} = \frac{t^{k-1}}{(k-1)!}e^{\lambda t}u(t).$$

Interpretation. Repeated poles introduce polynomial growth multiplying exponentials.

24 Complex-Conjugate Poles

If poles are

$$\lambda = \alpha \pm j\omega_0,$$

then the time response is

$$e^{\alpha t}(A\cos\omega_0 t + B\sin\omega_0 t)u(t).$$

- $\alpha < 0$: decaying oscillation.
- $\alpha = 0$: sustained oscillation.
- $\alpha > 0$: growing oscillation.

POLES, ZEROS, AND SYSTEM INSIGHT

Poles dominate long-term behavior. As $t \to \infty$, the pole with largest real part dominates.
Stability criterion (continuous time).

$$\boxed{\text{System stable} \iff \Re(\lambda_k) < 0\ \forall k}$$

Zeros shape the response. Zeros do not create modes but modify amplitudes and transient shape.
Pole-zero cancellation warning. Exact cancellation is fragile and rarely physical.

GEOMETRIC VIEW IN THE s-PLANE

- Vertical axis: oscillation ($j\omega$).
- Horizontal axis: growth/decay (σ).
- Poles near imaginary axis $\Rightarrow$ slow dynamics.
- Poles far left $\Rightarrow$ fast decay.

TLDR mental model.

- Poles define *what modes exist.*
- Zeros define *how strongly they appear.*
- ROC defines *which signal is physically realized.*