Ordinary Differential Equations

BASIC DEFINITIONS

ODE. An ordinary differential equation (ODE) is an equation of the form

$$F\left(x, y, y', ..., y^{(n)}\right) = 0,$$

where y is a single variable function depending on x, and $y', y'', ..., y^n$ are the k-order derivatives of y.

Example. Famous ODEs in Physics.

1. Newton's Law of Cooling: $\dfrac{dT}{dt} = -k(T - T_n)$
2. The Pendulum Equation: $\dfrac{d^2\theta}{dt^2} + \mu\dfrac{d\theta}{dt} + \omega_0^2 \sin(\theta(t)) = 0$
3. Damped Harmonic Oscillator: $m\dfrac{d^2x}{dt^2} + b\dfrac{dx}{dt} + kx = 0$
4. Rayleigh's Equation: $\dfrac{d^2y}{dx^2} + k\dfrac{dy}{dx} + m\left(\dfrac{dy}{dx}\right)^3 + n^2y = 0$

Linear ODE. A DE is called linear if it can be expressed as

$$P(y) = y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_1(x)y' + a_0(x)y = F(x),$$

Where $a_0(x), a_1(x), ..., a_{n-1}(x), F(x)$ are functions of x. This equation is called linear because

$$P(\alpha y_1 + \beta y_2) = \alpha P(y_1) + \beta P(y_2),\ \forall y_1, y_2 \in C^n(I).$$

In the example above, (1) and (3) are linear, while (2) and (4) are not.

Autonomous DE. An autonomous DE is a DE that does not explicitly depend on the independent variable. An n-order autonomous DE has the form

$$\frac{d^ny}{dx^n} = f(y) \quad [y = y(x)]$$

FIRST ORDER DIFFERENTIAL EQUATIONS

1 Basic Theory

First Order DE. A first-order DE is a DE that can be expressed as

$$y' = \frac{dy}{dx} = f(x, y).$$

Existence and Uniqueness. Consider $y' = f(x, y)$. Suppose there exists a rectangle

$$R = \{(x, y) \in \mathbb{R}^2 : a_1 \le x \le b_1,\ a_2 \le y \le b_2\} = [a_1, b_1] \times [a_2, b_2]$$

such that for any $(x, y) \in R$, both $f(x, y)$ and $f_y(x, y)$ are continuous. Then for any $(x_0, y_0) \in R\backslash\partial R$, the IVP $y' = f(x, y)$, $y(x_0) = y_0$ has a unique solution on some I containing x_0.

Singular Solution. A singular solution (from the word singularity) of an ODE, denoted y_s is a solution for which the IVP fails to have a unique solution at some point on the interval of solutions.

2 Separable Differential Equations

Standard Form. A DE is separable if it can be expressed as

$$\frac{dy}{dx} = f(x)g(y).$$

To solve this ODE, simply rearrange the equation and integrate both sides:

$$\int h(y)dy = \int f(x)dx$$

and we may obtain $H(y) = F(x) + C$, where $h = 1/g$, $H' = h$ and $F' = f$.

3 First Order Linear DE

Standard Form. A linear, first-order ODE is a differential equation that can be written in the form

$$\frac{dy}{dx} + p(x)y = q(x).$$

Integrating Factor. To solve this ODE, multiply both sides by the integrating factor $I(x) = e^{\int p(x)}dx$. By then, the equation becomes

$$\frac{d}{dx}[I(x)y] = I(x)q(x) \ \Rightarrow y = \frac{1}{I(x)}\left(\int I(x)q(x)\ dx + C\right)$$

4 Change of Variables and Homogeneous ODEs

Homogeneous Function. A function $f : U \subset \mathbb{R}^2 \to Y$ is said to be homogeneous of degree n if $f(tx, ty) = t^n f(x, y)$, $\forall t > 0$ and $(tx, ty) \in U$. This is the same as saying f can be written as

$$f(x, y) = x^n g(u) \quad [u = y/x]$$

or alternatively,

$$f(x, y) = y^n g(u) \quad [u = x/y]$$

Homogeneous ODEs. Given a differential equation $y' = f(x, y)$. This DE is homogeneous if $f(tx, ty) = f(x, y)$.

Substitution. To solve this ODE, divide by x (or y) and use the substitution $u = y/x$ (or x/y).

$$y = ux \Rightarrow \frac{dy}{dx} = \frac{du}{dx}x + u$$

Then the equation $y' = f(x, y)$ turns into

$$\frac{du}{dx}x + u = g(u) \Rightarrow \int \frac{du}{g(u) - u} = \int \frac{dx}{x}$$

5 Bernoulli Differential Equations

Standard Form. A Bernoulli differential equation is an equation that can be written in the form

$$\frac{dy}{dx} + p(x)y = q(x)y^n, \quad [n \ne 1]$$

Substitution. To solve this ODE, use the substitution $u = y^{1-n}$. By then,

$$\frac{du}{dx} = (1 - n)y^{-n}\frac{dy}{dx}.$$

Multiply both sides by $(1 - n)y^{-n}$ and obtain

$$\frac{du}{dx} + (1 - n)p(x)u = (1 - n)q(x).$$

This is a linear equation with integrating coefficient $I(x) = e^{(1-n)\int p(x)dx}$.

6 Exact Differential Equations

Definition. Given a DE $M(x, y)dx + N(x, y)dy = 0$, where M and N are continuous on some simply connected $D \subset \mathbb{R}^2$. This DE is called exact if there exists a function $f(x, y)$, called the potential function such that

$$f_x = M(x, y), \quad \text{and} \quad f_y = N(x, y).$$

Furthermore, $f(x, y) = C$ is the solution to this DE.

Test for Exactness. A DE $M(x, y)dx + N(x, y)dy = 0$ is exact if and only if $M_y = N_x$.

Method of Solving.

1. If it's easy, try to guess the solution.
2. If it's not easy, first find $\displaystyle\int M\ dx = m(x, y) + g(y) = f(x, y)$.
3. Find $f_y(x, y)$ and equate $f_y = N$ to find $g'(y) = N - m_y(x, y)$.
4. Find $g(y)$ from $g'(y)$ via integration.

Integrating Factors.

Sometimes, the equation $M(x, y)dx + N(x, y)dy = 0$ might not be exact, but it can be made exact by multiplying both sides by an integrating factor I.

1. If $P = (M_y - N_x)/N$ is a function of x only, then $I(x) = e^{\int P dx}$ is an integrating factor, and
$$I(x)M(x, y)dx + I(x)N(x, y)dy = 0$$
is exact.
2. If $Q = (N_x - M_y)/M$ is a function of y only, then $I(y) = e^{\int Q dy}$ is an integrating factor, and
$$I(y)M(x, y)dx + I(y)N(x, y)dy = 0$$
is exact.

7 Linear Coefficient Differential Equations

Standard Form. A Linear Coefficient DE is a DE of the form

$$(a_1x + b_1y + c_1)dx + (a_2x + b_2y + c_2)dy = 0, \tag{1}$$

where the coefficients of dx and dy, called d_1 and d_2 are linear, non-homogeneous, and represent lines.

Remark. If $c_1 = c_2 = 0$, then the DE above is an equation with homogeneous coefficients which can be solved by the method of Homogeneous Equation.

CASE 1: d_1 and d_2 are non-parallel.

Method 1: Translation of Coordinate. Since $d_1 \nparallel d_2$,

$$d_1(x, y) = a_1x + b_1y + c_1 = 0$$
$$d_2(x, y) = a_2x + b_2y + c_2 = 0$$

have a unique solution for x and y. Call this unique point (h, k). We now translate the origin to (h, k), by letting

$$x = \overline{x} + h, \quad \text{and} \quad y = \overline{y} + k$$

then the DE above becomes, with respect to this new origin $(\overline{0}, \overline{0})$,

$$[a_1(\overline{x} + h) + b_1(\overline{y} + k) + c_1]d\overline{x} + [a_2(\overline{x} + h) + b_2(\overline{y} + k) + c_2]d\overline{y} = 0,$$
$$\Rightarrow [a_1\overline{x} + b_1\overline{y} + \underbrace{(a_1h + b_1k + c_1)}_{d_1(h,k)=0}]d\overline{x} + [a_2\overline{x} + b_2\overline{y} + \underbrace{(a_2h + b_2k + c_2)}_{d_2(h,k)=0}]d\overline{y} = 0,$$
$$\Rightarrow (a_1\overline{x} + b_1\overline{y})d\overline{x} + (a_2\overline{x} + b_2\overline{y})d\overline{y} = 0,$$

which is now a homogeneous type solvable for $\overline{x}$ and $\overline{y}$.

Method 2: u, v Substitution. Another method to solve equations of this type is by substitution. Let

$$\begin{cases} u = a_1x + b_1y + c_1 \\ v = a_2x + b_2y + c_2 \end{cases} \Rightarrow \begin{cases} du = a_1dx + b_1dy \\ dv = a_2dx + b_2dy \end{cases}$$

Now solve the equation above for dx and dy, we obtain

$$dx = \frac{b_2du - b_1dv}{a_1b_2 - b_1a_2} \quad \text{and} \quad dy = \frac{a_1dv - a_2du}{a_1b_2 - b_1a_2}$$

Substitute this back to equation (1), we obtain

$$u\frac{b_2du - b_1dv}{a_1b_2 - b_1a_2} + v\frac{a_1dv - a_2du}{a_1b_2 - b_1a_2} = 0,$$
$$\Rightarrow (b_2u - a_2v)du + (a_1v - b_1u)dv = 0.$$

This equation is homogeneous, now solve by letting $t = u/v$.

CASE 2: d_1 and d_2 Are Parallel. The line d_2 is parallel to d_1 if it is of the form

$$\begin{matrix} d_1(x, y) = a_1x + b_1y + c_1 = 0 \\ d_2(x, y) = \alpha a_1x + \alpha b_1y + c_2 = 0 \end{matrix}, \quad c_1 \neq c_2.$$

In this case, let $u = a_1x + b_1y + c_1 \Rightarrow du = a_1dx + b_1dy$, or,

$$dx = \frac{du - b_1dy}{a_1}$$

and let $c_2 = \alpha c_1 + k$. Substitute back in the equation

$$(a_1x + b_1y + c_1)dx + (a_2x + b_2y + c_2)dy = 0,$$

we obtain

$$u\left(\frac{du - b_1dy}{a_1}\right) + (\alpha u + k)dy = 0,$$

which simplifies to

$$udu + ((\alpha a_1 - b_1)u + ka_1)dy = 0.$$

This is a separable equation and can be solved easily.

CASE 3: d_1 and d_2 Are Coincident. Clearly, if d_2 is coincident to d_1, then d_2 has the form

$$d_2(x, y) = \alpha d_1(x, y) = \alpha(a_1x + b_1y + c_1).$$

The linear coefficient DE becomes

$$(a_1x + b_1y + c_1)dx + \alpha(a_1x + b_1y + c_1)dy = 0.$$

Note that the function defined by $a_1x + b_1y + c_1 = 0$ clearly satisfies the DE. We count this as a particular solution. Assume that $a_1x + b_1y + c_1 \neq 0$. Then, divide both sides by $a_1x + b_1y + c_1$ and we obtain

$$dx + \alpha dy = 0,$$

and therefore its complementary solution is

$$x + \alpha y = C.$$

8 Riccati Differential Equation

Standard Form. Riccati equation is a DE that can be written in the form

$$y' = p(x) + y\, q(x) + y^2 s(x).$$

Solution. If $s(x) = 0$, then

$$y' = p(x) + q(x)y$$

is just a first-order linear DE. If $p(x) = 0$, then

$$y' = yq(x) + y^2 s(x)$$

is just a Bernoulli differential equation. Let $t = s(x)y$, where $s(x) \neq 0$. Then, $t' = s'(x)y + y's(x)$. Substitute y' back in,

$$t' = s'y + s(p + y\, q + y^2 s) = t\frac{s'}{s} + sp + qt + t^2$$
$$\Rightarrow t' = t^2 + t\left(\frac{s'}{s} + 1\right) + sp$$

Let $P(x) = \left(\frac{s'}{s} + 1\right)$, and $G(x) = sp$. Then,

$$t' - t^2 = tP(x) + Q(x)$$

Also, let

$$t = \frac{\varphi'}{\varphi} \quad \Rightarrow \quad t' = \frac{-\varphi''}{\varphi} + \left(\frac{\varphi'}{\varphi}\right)^2 = -\frac{\varphi''}{\varphi} + t^2 \quad \Rightarrow \quad t' - t^2 = -\frac{\varphi''}{\varphi}$$

Substitute this back in the equation right above this one,

$$-\frac{\varphi''}{\varphi} = tP(x) + Q(x) = -\frac{\varphi'}{\varphi}P(x) + Q(x)$$

This implies

$$\varphi'' - \varphi' P(x) + Q(x)\varphi = 0.$$

This is a homogeneous, second-order linear DE. Recall that $y = \frac{t}{s} = \frac{-\varphi'}{s\varphi}$, so if we can find φ, then we can find y.

9 d'Alembert-Lagrange Differential Equation

Standard Form. The DE of the form

$$y = xf(y') + g(y'), \tag{2}$$

where $f(y')$ and $g(y')$ are differentiable, known function on an interval I is called the Lagrange Differential Equation.

Solution. Let $y' = p(x)$. Differentiate both sides of (2), we obtain

$$p = xf'(p)p' + f(p) + g(p)p' \quad \Rightarrow \quad p - f(p) = [xf'(p) + g'(p)]\frac{dp}{dx}$$

If we interchange the role of x and p by treating x as a dependent variable on p; that is, $x = x(p)$. Thus, from the non-linear equation right above,

$$\frac{dx}{dp} - \frac{f'(p)}{p - f(p)}x = \frac{g'(p)}{p - f(p)},$$

provided that $f(p) \neq p$. By solving the equation above, the solution of 2 is thereby obtained in parametric form:

$$\begin{cases} x = x(p) \\ y = x(p)f(p) + g(p) \end{cases}$$

The case where $f(p) - p = 0$, the equation (2) has a particular solution of the form

$$y = xf(c) + g(c),$$

where c is the solution to the equation $f(p) - p = 0$.

10 Clairaut Differential Equation

Standard Form. The DE of the form

$$y = xy' + f(y'),$$

where $f(y')$ is a nonlinear, continuously differentiable function, is called the Clairaut Differential Equation.

Solution. To solve this DE, first, differentiate both sides with respect to x

$$y' = y' + xy'' + y''f'(y') \Longrightarrow \left[x + f'(y')\right]y'' = 0$$

This implies either $x + f'(y') = 0$, or $y'' = 0$.

- Case 1: $y'' = 0$. This implies $y' = C$. Substitute this back into the DE, we obtain a family of solutions of
$$y = Cx + f(C)$$
- Case 2: $x + f'(y') = 0$. This case defines one singular solution $y(x)$, whose graph is the envelope of graphs of the general solutions. For this equation, the singular solution is represented as $(x(p), y(p))$, where $p = y'$. The parametric description of the singular solution is
$$\mathbf{r}(t) = (x(t), y(t)) = \left(-f'(t),\ f(t) - tf'(t)\right), \quad t \in \mathbb{R}.$$

11 Chrystal's Differential Equation

Standard Form. Chrystal's equation is a first-order nonlinear DE of the form

$$(y')^2 + Axy' + By + Cx^2 = 0.$$

where $y = y(x)$.

Solution. Let $4By = (A^2 + 4C - z^2)x^2$. Then,

$$xz\frac{dz}{dx} = A^2 + AB - 4C \pm Bz - z^2$$

The equation is now separable, thus

$$\int \frac{z\,dz}{A^2 + AB - 4C \pm Bz - z^2} = \int \frac{dx}{x}$$

let a, b be the root of the equation $A^2 + AB - 4C \pm Bz - z^2 = 0$. We can solve for a and b:

$$\begin{aligned} a,\ b &= \frac{-(\pm B) \pm \sqrt{B^2 - 4(-1)(A^2 + AB - 4C)}}{2(-1)} \\ &= \frac{\pm B \mp \sqrt{4A^2 + 4AB + B^2 - 16C}}{2} \\ &= \pm \frac{B - \sqrt{(2A + B)^2 - 16C}}{2} \end{aligned}$$

- Case 1: $a \neq b$. The integral becomes

$$\int \frac{z\,dz}{(z-a)(z-b)} = \int \frac{dx}{x}$$

$$\begin{aligned} &\Rightarrow \int \frac{a}{a-b}\frac{1}{z-a} - \frac{b}{a-b}\frac{1}{z-b}dz = \ln|x| + C \\ &\Rightarrow \ln\left|\frac{(z-a)^{a/(a-b)}}{(z-b)^{b/(a-b)}}\right| = \ln|kx|, \quad k = \ln|C| \\ &\Rightarrow x\frac{(z-a)^{a/(a-b)}}{(z-b)^{b/(a-b)}} = k \end{aligned}$$

  is the solution.

- Case 2: $a = b$. The integral becomes

$$\int \frac{z\,dz}{(z-a)^2} = \int \frac{dx}{x}$$

  integrate both sides gives

$$-\frac{z}{z-a} + \ln|z-a| = \ln|Kx|, \quad \Rightarrow \quad \ln\left|\exp\left(\frac{z}{a-z}\right)(z-a)\right| = \ln|kx|$$

$$\Rightarrow \quad x(z-a)\exp\left(\frac{z}{a-z}\right) = k$$

  is the solution.

When one of the roots is zero, i.e., when $A^2 + AB - 4C = 0$, the equation reduces to Clairaut's differential equation, and a parabolic solution is obtained:

$$x(z \pm B) = k, \quad \Rightarrow \quad 4By = -ABx^2 - (k \pm Bx)^2.$$

The above family of parabolas is enveloped by the parabola $eBy = -ABx^2$. Therefore, this enveloping parabola is a singular solution.

12 Solving Differential Equations Using Substitution

Just like integration, various strange, but brilliant substitutions can be introduced to solve differential equations.

Example. Solve

$$\frac{dy}{dx} = (-2x + y)^2 - 7, \quad y(0) = 0$$

Solution. Let $u = -2x + y$, then $du/dx = -2 + dy/dx$, so the differential equation becomes

$$\frac{du}{dx} + 2 = u^2 - 7, \quad \Rightarrow \quad \frac{du}{dx} = u^2 - 9,$$

which can easily be solved using the method of separation.

THEORY OF INITIAL VALUE PROBLEM

Lipschitz Condition. A function $f(x, y)$ is said to satisfy a Lipschitz condition in the variable y on a set $D \subset \mathbb{R}^2$ if there exists a constant $L > 0$ such that

$$|f(x, y_1) - f(x, y_2)| < L|y_1 - y_2|$$

whenever (x, y_1) and (x, y_2) are in D. The constant L is called a Lipschitz constant for f.

Convex Set. A set $D \subset \mathbb{R}^2$ is said to be convex if whenever (x_1, y_1) and (x_2, y_2) belong to D, then

$$((1-\lambda)x_1 + \lambda x_2,\ (1-\lambda)y_1 + \lambda y_2) \in D, \quad \forall \lambda \in [0, 1].$$

Theorem 1. Suppose $f(x, y)$ is defined on a convex set $D \subset \mathbb{R}^2$. If there exists a constant $L > 0$ such that

$$|f_y(x, y)| \leq L, \quad \forall (x, y) \in D,$$

then f satisfies a Lipschit condition on D in the variable y with Lipschitz constant L.

Theorem 2. Suppose that $D = \{(x, y) \in \mathbb{R}^2 \mid a \leq x \leq b,\ -\infty < y < \infty\}$, and that $f(x, y)$ is continuous on D. If f satisfies a Lipschitz condition on D in the variable y, then the IVP

$$y' = f(x, y), \quad a \leq x \leq b, \quad y(a) = y_0$$

has a unique solution y for $a \leq x \leq b$.

Theorem 3. Consider $y' = f(x, y)$. Suppose there exists a rectangle

$$R = \{(x, y) \in \mathbb{R}^2 : a_1 \leq x \leq b_1,\ a_2 \leq y \leq b_2\} = [a_1, b_1] \times [a_2, b_2]$$

such that for any $(x, y) \in R$, both $f(x, y)$ and $f_y(x, y)$ are continuous. Then for any $(x_0, y_0) \in R \backslash \partial R$, the IVP $y' = f(x, y)$, $y(x_0) = y_0$ has a unique solution on some I containing x_0.

13 Picard Iteration (Successive Approximations)

Motivation. Picard iteration provides a constructive method for proving the existence and uniqueness of solutions to the IVP $y' = f(x, y)$, $y(x_0) = y_0$. It also gives us a way to approximate the solution.

Integral Form of the IVP. The IVP $y' = f(x, y)$, $y(x_0) = y_0$ is equivalent to the integral equation

$$y(x) = y_0 + \int_{x_0}^{x} f(t, y(t))\,dt.$$

The Iteration. Starting from the initial guess $\phi_0(x) = y_0$, we define the Picard iterates recursively by

$$\phi_{n+1}(x) = y_0 + \int_{x_0}^{x} f(t, \phi_n(t))\,dt, \quad n = 0, 1, 2, \ldots$$

Under the hypotheses of the existence and uniqueness theorem (i.e., f and f_y are continuous in a neighborhood of (x_0, y_0)), the sequence $\{\phi_n\}$ converges uniformly to the unique solution $y(x)$ on some interval $|x - x_0| \leq h$.

Example. Solve $y' = y$, $y(0) = 1$ using Picard iteration.
Solution. Here $f(x, y) = y$, $x_0 = 0$, $y_0 = 1$.

$$\begin{aligned} \phi_0(x) &= 1 \\ \phi_1(x) &= 1 + \int_0^x 1\,dt = 1 + x \\ \phi_2(x) &= 1 + \int_0^x (1 + t)\,dt = 1 + x + \frac{x^2}{2} \\ \phi_3(x) &= 1 + \int_0^x \left(1 + t + \frac{t^2}{2}\right)dt = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} \end{aligned}$$

In general, $\phi_n(x) = \sum_{k=0}^{n} \frac{x^k}{k!}$, which converges to $y(x) = e^x$.

Example. Apply Picard iteration to $y' = x + y$, $y(0) = 1$.
Solution. Here $f(x, y) = x + y$.

$$\begin{aligned} \phi_0(x) &= 1 \\ \phi_1(x) &= 1 + \int_0^x (t + 1)\,dt = 1 + x + \frac{x^2}{2} \\ \phi_2(x) &= 1 + \int_0^x \left(t + 1 + t + \frac{t^2}{2}\right)dt = 1 + x + x^2 + \frac{x^3}{6} \\ \phi_3(x) &= 1 + \int_0^x \left(t + 1 + t + t^2 + \frac{t^3}{6}\right)dt = 1 + x + x^2 + \frac{x^3}{3} + \frac{x^4}{24} \end{aligned}$$

The exact solution is $y(x) = 2e^x - x - 1$.

Error Bound. If $|f_y| \leq L$ in D and $|f| \leq M$ in D, then

$$|y(x) - \phi_n(x)| \leq \frac{ML^n}{(n+1)!}|x - x_0|^{n+1}$$

for $|x - x_0| \leq h$, where $h = \min\left(a, \frac{b}{M}\right)$ and a, b define the rectangle around (x_0, y_0).

14 Numerical Methods for ODEs

Given the IVP $y' = f(x,y)$, $y(x_0) = y_0$, we want to approximate the solution at equally spaced points $x_n = x_0 + nh$, where h is the step size. Denote the approximation at x_n by y_n.

Euler's Method (Forward Euler). The simplest numerical method. Approximate y' by a forward difference:

$$y_{n+1} = y_n + hf(x_n, y_n).$$

This is a first-order method: the local truncation error (LTE) is $O(h^2)$ and the global error is $O(h)$.

Backward Euler (Implicit Euler).

$$y_{n+1} = y_n + hf(x_{n+1}, y_{n+1}).$$

This is an implicit method (requires solving for y_{n+1}), but it has better stability properties than forward Euler. Also first-order.

Improved Euler (Heun's Method). A second-order predictor-corrector method:

$$\begin{aligned} k_1 &= f(x_n, y_n) && \text{(predictor slope)} \\ k_2 &= f(x_n + h,\ y_n + hk_1) && \text{(corrector slope)} \\ y_{n+1} &= y_n + \frac{h}{2}(k_1 + k_2) \end{aligned}$$

The LTE is $O(h^3)$ and the global error is $O(h^2)$.

Midpoint Method. Another second-order method:

$$y_{n+1} = y_n + hf\left(x_n + \frac{h}{2},\ y_n + \frac{h}{2}f(x_n, y_n)\right).$$

Classical Runge-Kutta (RK4). The most widely used single-step method. Fourth-order accuracy:

$$\begin{aligned} k_1 &= f(x_n,\ y_n) \\ k_2 &= f\left(x_n + \tfrac{h}{2},\ y_n + \tfrac{h}{2}k_1\right) \\ k_3 &= f\left(x_n + \tfrac{h}{2},\ y_n + \tfrac{h}{2}k_2\right) \\ k_4 &= f(x_n + h,\ y_n + hk_3) \\ y_{n+1} &= y_n + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4). \end{aligned}$$

The LTE is $O(h^5)$ and the global error is $O(h^4)$.

Summary of Methods.

Method	Order	Global Error
Euler (Forward)	1	$O(h)$
Backward Euler	1	$O(h)$
Improved Euler / Heun	2	$O(h^2)$
Midpoint	2	$O(h^2)$
RK4	4	$O(h^4)$

Example. Use Euler's method with $h = 0.1$ to approximate $y(0.2)$ for $y' = x + y$, $y(0) = 1$.
Solution.

$$\begin{aligned} y_1 &= y_0 + hf(x_0, y_0) = 1 + 0.1(0 + 1) = 1.1 \\ y_2 &= y_1 + hf(x_1, y_1) = 1.1 + 0.1(0.1 + 1.1) = 1.22 \end{aligned}$$

The exact value is $y(0.2) = 2e^{0.2} - 0.2 - 1 \approx 1.2428$.

SECOND AND HIGHER ORDER DIFFERENTIAL EQUATIONS

15 General Theory and Tools

Notations.

1. **The Differential Operator.** Let x be the independent variable, and y be the dependent variable. Then, the first and n-th order differential operator on an interval I, $D^n : C^n(I) \to C^{n-1}(I)$ is defined, respectively by
$$D(y) = \frac{dy}{dx} \quad \text{and} \quad D^n(y) = \frac{d^n y}{dx^n}.$$
Overall, D^n is a linear transformation.
2. **The Linear Operator.** An n-th order linear DE can be expressed as
$$[D^n a_{n-1}(x)D^{n-1} + ... + a_1(x)D + a_0(x)]y = F(x).$$
Using the Differential operator, define L to be
$$L = D^n a_{n-1}(x)D^{n-1} + ... + a_1(x)D + a_0(x),$$
and hence a n-th order linear DE can be expressed as $Ly = F(x)$. Consequently, L is also a linear transformation.
3. **The Polynomial Operator.** If $a_{n-1}, a_{n-2}, ..., a_1, a_0$ are constants, then we write
$$P(D) = D^n + a_{n-1}D^{n-1} + ... + a_1 D + a_0$$
for $P(D)y = F(x)$. This is the same notation as the L operator, but the coefficients are constants.

The Wronskian and Linearly Independence of the Solutions.

1. **The Wronskian.** Let $f_1, f_2, ..., f_n$ be $n-1$ times differentiable function defined on the interval I, that is, $f_1, f_2, ..., f_n \in C^{n-1}(I)$. Then, the Wronskian of $f_1, f_2, ..., f_n$, written $W(f_1, f_2, ..., f_n)(x)$ is a determinant defined by
$$W(f_1, f_2, ..., f_n)(x) = \det \begin{bmatrix} f_1 & f_2 & \cdots & f_n \\ f_1' & f_2' & \cdots & f_n' \\ \vdots & \vdots & \ddots & \vdots \\ f_1^{(n-1)} & f_2^{(n-1)} & \cdots & f_n^{(n-1)} \end{bmatrix}$$
2.
Theorem 4. Let $V = C^{n-1}(I)$ be a vector space, and let $f_1, f_2, ..., f_n \in V$. If $W(f_1, f_2, ..., f_n)(x) \neq 0$ at some $x_0 \in I$, then $S = \{f_1, f_2, ..., f_n\}$ is LI.

Note that the converse of this theorem is not true. That is, it is not true that if $W = 0$, then S is LD. If $W = 0$, then the theorem fails. However, the converse would be true instead if $f_1, f_2, ..., f_n$ were solutions of some differential equations.

Existence and Uniqueness of n^{th} order Linear DE. Let I be an open interval. If $a_0(x), a_1(x), ..., a_{n-1}(x), F(x)$ are continuous on I, then for any $x_0 \in I$, the IVP

$$\begin{cases} Ly = F(x) \\ y(x_0) = y_0,\ y'(x_0) = y_1, \cdots, y^{(n-1)}(x_0) = y_{n-1} \end{cases}$$

has a unique solution. That is, there exists only one function y that satisfies the condition above.

Theorem 5. Let $L = D^n a_{n-1}(x)D^{n-1} + ... + a_1(x)D + a_0(x)$. If $a_{n-1}(x), a_{n-2}(x), \ldots, a_1(x), a_0(x)$ are continuous on some interval I, then

$$S = \{y \in C^n(I) \mid Ly = 0\},$$

the set of all solutions to the DE is a vector space and $\dim S = n$.

Complementary and Particular Solutions.

- Let y_p be any solution to the non-homogeneous equation $Ly = F(x)$, then y_p is called a particular solution of a DE.
- Let y_c be a family of solutions to $Ly = 0$, then y_c is called the complementary solution of the DE.

Theorem 6. All solutions to the DE $Ly = F(x)$ are given by $y = y_c + y_p$

Theorem 7 (Principle of Superposition). If $\{y_1, y_2, \ldots, y_n\}$ are solution to the DE $Ly = 0$, then $C_1y_1 + C_2y_2 + \ldots + C_ny_n$ must also be the solution.

16 Constant Coefficient Second Order DE

Characteristic Polynomial. Given a DE $y'' + ay' + by = 0$, where a and b are constants. Substituting $y = Ae^x$ and its first two derivatives to the equation, and divide both sides by Ae^x (since $e^x \neq 0, \forall x \in \mathbb{C}$) gives the equation

$$r^2 + ar + b = 0,$$

called the characteristic polynomial. It can be seen that the DE $P(D)y = 0$ is equivalent to the characteristic polynomial $P(r)$.

Another important thing to remember is that the equation

$$r^2 - 2ar + a^2 + b^2 = 0$$

always has two complex solutions $r = a \pm bi$.

Homogeneous Equations. Given a linear equation $P(D)y = 0$, where $\deg(P) = 2$. First, solve for the equation $P(r) = 0$.

1. **Case 1: Two Real Solutions.** If r_1 and r_2 are real solutions, then $\{e^{r_1 x}, e^{r_2 x}\}$ is a basis for S, and the solution is
$$y = C_1e^{r_1 x} + C_2e^{r_2 x}.$$
If $P(r) = 0$ has two real solution of the form $r = \pm p$, then, the solution can also be expressed as
$$y = C_1 \cosh(px) + C_2 \sinh(px)$$

2. **Case 2: One Real Solution.** If r is a real solution, then $\{e^{rx}, xe^{rx}\}$ is a basis for S, and the solution to the DE is
$$y = C_1 e^{r_1 x} + C_2 x e^{rx}.$$
3. **Case 3: Two Complex Solutions.** If $a \pm bi$ are complex solutions to $p(r)$, then $\{e^{(a+bi)x}, e^{(a-bi)x}\}$ is a basis for S. Express this in trigonometric form, we have $\{e^{ax}\cos bx, e^{ax}\sin bx\}$ is a basis for S, and the solution to the DE is
$$y = C_1 e^{ax}\cos bx + C_2 e^{ax}\sin bx.$$

Theorem 8. Given a homogeneous DE $(D-a)^n y = 0$. Then, the set of solutions to this DE is a vector space of dimension n, and
$$\{e^{ax}, xe^{ax}, x^2 e^{ax}, ..., x^{n-1}e^{ax}\}$$
is a basis for the set of solution S.

In general, given a homogeneous DE $[P(D)]^n y = 0$. Let $r_1, r_2, ..., r_n$ be solutions to $P(r) = 0$, where $r_i \neq r_j$, for any $1 \leq i \leq j \leq n$. Then,
$$S = \bigcup_{k=0}^{n-1} \{x^k e^{r_1 x}, \dots, x^k e^{r_n x}\}$$
$$= \{e^{r_1 x}, ..., e^{r_n x}\} \cup \{xe^{r_1 x}, ..., xe^{r_n x}\} \cup \ldots \cup \{x^{n-1}e^{r_1 x}, ..., x^{n-1}e^{r_n x}\}$$
is a basis for the set of solutions of this DE.

Example. Solve the DE
$$(D^2 + 2D + 10)^2 (D^2 + 1)^3 (D - 1) y = 0$$
Solution. The characteristic polynomial of the DE is
$$(r^2 + 2r + 10)^2 (r^2 + 1)^3 (r - 1) = 0,$$
which gives
$$r = -1 \pm 3i, \; r = 0 \pm i \;\text{ and }\; r = 1$$
Since this is an 11-th-order differential equation, the solution space's dimension must also be 11. This means that there are 11 linearly independent solutions to this DE, and the general solution is given by
$$y = C_1 e^{-x}\cos 3x + C_2 e^{-x}\sin 3x + C_3 x e^{-x}\cos 3x + C_4 x e^{-x}\sin 3x + C_5 \cos x$$
$$+ C_6 \sin x + C_7 x\cos x + C_8 x \sin x + C_9 x^2 \cos x + C_{10} x^2 \sin 3x + C_{11} e^x$$

17 Method of Undetermined Coefficient

Strategy. The method of undetermined coefficient requires us to guess the form of the particular solution.
To solve the nonhomogeneous ODE
$$P(D)y = F(x),$$

1. Find the complementary solution y_c by solving $P(D)y = 0$.
2. Guess the form of y_p, then find the constants by plugging y_p in the DE.
3. The general solution is given by $y = y_c + y_p$.

The following table is a summary of how we can guess the particular solution.

RHS	**If RHS is not part of y_c**	**If RHS is part of y_c**
Ce^{ax}	$y_p = Ae^{ax}$	$y_p = Axe^{ax}$
$\sin bx$, $\cos bx$ or both	$y_p = A\cos ax + B\sin ax$	$y_p = x(A\cos ax + B\sin ax)$
Polynomial of degree n	$y_p = a_0 + a_1 x + \ldots + a_n x^n$	$y_p = x(a_0 + \ldots + a_n x^n)$

Example. Solve the DE
$$y'' - 4y' + 3y = 5e^{4x}$$
Solution. The particular solution of the DE is
$$y_c = C_1 e^x + C_2 e^{3x}$$
The right hand side of the equation is $5e^{4x}$, and it is not part of y_c, therefore, y_p might look something like
$$y_p = Ae^{4x} \Longrightarrow y_p' = 4Ae^{4x} \Longrightarrow y_p'' = 16Ae^{4x}$$
Substitute this back into the DE, and simplify e^{4x}:
$$16Ae^{4x} - 4(4Ae^{4x}) + 3(Ae^{4x}) = 5e^{4x} \Longrightarrow A = \frac{5}{3}$$
Therefore, the general solution is
$$y = y_c + y_p = C_1 e^x + C_2 e^{3x} + \frac{5}{3}e^{4x}$$

18 Method of Annihilators

Annihilators. Let D be the differential operator, $Q(D)$ be the polynomial operator. We say $Q(D)$ annihilates y if $Q(D)y = 0$. The polynomial of the smallest degree that annihilates y is called the annihilator of y.

Examples.

- $D - 3$ annihilates e^{3x} because $(D-3)e^{3x} = De^{3x} - 3e^{3x} = 0$.
- $(D-3)^3$ annihilates $x^2 e^{3x}$.
- $D^2 + 9$ annihilates $\cos 3x$ and $\sin 3x$.
- $(D^2 + 9)^4$ annihilates $x^3 \cos 3x$ and $x^3 \sin 3x$.
- D^4 annihilates $x^3 + x^2 + 1$.

Common Annihilators.

Function	**Annihilators**
$x^n e^{ax}$	$(D-a)^{n+1}$
$x^n \cos bx$ or $x^n \sin bx$	$(D^2 + b^2)^{n+1}$
$x^n e^{ax}\cos bx$ or $x^n e^{ax}\sin bx$	$(D^2 - 2aD + a^2 + b^2)^{n+1}$
Polynomial of deg n	D^{n+1}

Theorem 9. Given that $P(D)$ annihilates y_1, and $Q(D)$ annihilates y_2. Then,

1. If $P(D) = Q(D)$, then $P(D)$ annihilates $y_1 + y_2$.
2. If $P(D) \neq Q(D)$, then $\text{lcm}(P(D), Q(D))$ annihilates $y_1 + y_2$.
3. Special Case: If $\text{gcf}(P(D), Q(D)) = 1$ (coprime), then $P(D)Q(D)$ annihilates $y_1 + y_2$.

Example. Find the annihilator of
$$6x^3 e^{5x} + 4x^3 + 2x^2 + 5.$$
There are two ways to do this:
$$(6x^3 e^{5x}) + (4x^3 + 2x^2 + 5) \quad \text{or} \quad (6x^3 e^{5x}) + (4x^3) + (2x^2) + (5).$$
First way:
$$(\underbrace{6x^3 e^{5x}}_{(D-5)^4}) + (\underbrace{4x^3 + 2x^2 + 5}_{D^4})$$
Therefore, the annihilator of this function is $(D-5)^4 D^4$.

Second way:
$$(\underbrace{6x^3 e^{5x}}_{(D-5)^4}) + (\underbrace{4x^3}_{D^4}) + (\underbrace{2x^2}_{D^3}) + (\underbrace{5}_{D})$$
Since $\text{gcf}(D^4, D^3, D) = D^4$, therefore the annihilator of this function is $(D-5)^4 D^4$.

Solution. Given a linear, constant coefficient ODE $P(D)y = F(x)$. Assume that $Q(D)$ annihilates $F(x)$. "Multiply" both sides by $Q(D)$, we obtain
$$Q(D)P(D)y = Q(D)F(x) = 0.$$
The equation $Q(D)P(D)y = 0$ is a linear, homogeneous DE, and can be solved quite easily.

To find the form of y_p. First, solve the DE $P(D)y = 0$ to find y_c. Then, solve the equation $Q(D)P(D)y = 0$ to find y, the general solution. Then, simply remove y_c out of y to obtain y_p. Since y_p is a particular solution, replace y_p back into the DE $P(D)y = F(x)$ to find and solve for the actual values of the coefficients.

Example. Solve the differential equation
$$y'' + 3y' + 2y = 3x^2 + 2e^{-x}$$
Solution. Write this in differential form:
$$(D^2 + 3D + 2)y = (D+1)(D+2)y = 3x^2 + 2e^{-x}$$
The complementary solution can be obtained easily from the homogeneous case
$$y_c = Ae^{-x} + Be^{-2x}$$
The annihilator of $3x^2$ is D^3, and of $2e^{-x}$ is $D + 1$, therefore the annihilator of $3x^2 + 2e^{-x}$ is $D^3(D+1)$. Differentiate both sides using the obtained annihilator, we have
$$D^3(D+1)^2(D+2)y = 0$$
therefore the solution to this equation is
$$y = C_1 + C_2 x + C_3 x^2 + C_4 e^{-x} + C_5 x e^{-x} + C_6 e^{-2x},$$
which gives us the particular solution of
$$y_p = C_1 + C_2 x + C_3 x^2 + C_5 x e^{-x}$$

Take the first and second derivatives of y_p, and substitute them back in the DE to solve for the constants, we obtain

$$y_p = \frac{21}{4} - \frac{9}{2}x + \frac{3}{2}x^2 + 2xe^{-x}$$

Therefore, the general solution for this DE is

$$y = Ae^{-x} + Be^{-2x} + \frac{21}{4} - \frac{9}{2}x + \frac{3}{2}x^2 + 2xe^{-x}$$

19 Method of Order Reduction

Overview. The method of order reduction (also called reduction of order) is used to find a second linearly independent solution y_2 of a second-order linear homogeneous DE when one solution y_1 is already known.

Setup. Given a DE in standard form

$$y'' + p(x)y' + q(x)y = 0,$$

and one known solution $y_1(x) \neq 0$. Assume the second solution has the form

$$y_2 = v(x)y_1(x),$$

where $v(x)$ is an unknown function to be determined.

Derivation. Compute the derivatives:

$$\begin{aligned} y_2' &= v'y_1 + vy_1', \\ y_2'' &= v''y_1 + 2v'y_1' + vy_1''. \end{aligned}$$

Substitute into the DE:

$$\begin{gathered} (v''y_1 + 2v'y_1' + vy_1'') + p(x)(v'y_1 + vy_1') + q(x)vy_1 = 0 \\ v''y_1 + v'(2y_1' + py_1) + v\underbrace{(y_1'' + py_1' + qy_1)}_{=0} = 0 \end{gathered}$$

Since y_1 is a solution, the last group vanishes. Dividing by y_1:

$$v'' + \left(\frac{2y_1'}{y_1} + p(x)\right)v' = 0.$$

This is a first-order DE in $w = v'$:

$$w' + \left(\frac{2y_1'}{y_1} + p(x)\right)w = 0,$$

which is separable. Solving:

$$w(x) = v'(x) = \frac{1}{[y_1(x)]^2}\exp\left(-\int p(x)\,dx\right).$$

Integrating once more:

$$v(x) = \int \frac{1}{[y_1(x)]^2}\exp\left(-\int p(x)\,dx\right)dx.$$

Therefore, the second solution is

$$\boxed{y_2(x) = y_1(x)\int \frac{1}{[y_1(x)]^2}\exp\left(-\int p(x)\,dx\right)dx.}$$

Example. Given that $y_1 = e^x$ is a solution to $y'' - 2y' + y = 0$, find y_2.
Solution. Here $p(x) = -2$. Using the formula:

$$v(x) = \int \frac{1}{e^{2x}}\exp\left(-\int(-2)\,dx\right)dx = \int \frac{e^{2x}}{e^{2x}}\,dx = \int dx = x.$$

Therefore, $y_2 = xe^x$, and the general solution is $y = C_1e^x + C_2xe^x$.

Example. Given that $y_1 = x$ is a solution to $x^2y'' - xy' + y = 0$ for $x > 0$, find y_2.
Solution. Rewrite in standard form: $y'' - \frac{1}{x}y' + \frac{1}{x^2}y = 0$, so $p(x) = -1/x$.

$$v(x) = \int \frac{1}{x^2}\exp\left(-\int \frac{-1}{x}\,dx\right)dx = \int \frac{1}{x^2}\cdot x\,dx = \int \frac{1}{x}\,dx = \ln|x|.$$

Therefore, $y_2 = x\ln x$, and the general solution is $y = C_1x + C_2x\ln x$.

Reducing Higher-Order DEs. For an n-th order DE, if one solution y_1 is known, the substitution $y = vy_1$ reduces the order by one. Repeated application can reduce an n-th order DE to a first-order DE if $n-1$ solutions are known.

20 Variation of Parameters Method (VoP)

20.1 Derivation of Variation of Parameters Formula. (Frobenius Method)

Given a linear DE $Ly = F(x)$, where $L = D^n a_{n-1}(x)D^{n-1} + ... + a_1(x)D + a_0(x)$. Here, the L operator indicates that the coefficients $a_{n-1}, \dots, a_1, a_0$ are not necessarily constants. First, we need to find the solution to the homogeneous DE

$$Ly = 0.$$

This DE's solution basis is $\{y_1, y_2, \dots, y_n\}$, and the general, complementary solution is

$$y_c = C_1y_1 + C_2y_2 + \dots + C_ny_n.$$

Assume that there exists a particular solution of the form

$$y_p = u_1y_1 + u_2y_2 + \dots + u_ny_n,$$

where $u_1, u_2, \dots, u_n$ are functions of x.

The goal here is to find what $u_1, u_2, \dots, u_n$ are. Taking the first derivative of y_p, we obtain

$$y_p' = (u_1y_1' + \dots + u_ny_n') + (u_1'y_1 + \dots + u_n'y_n).$$

Since $u_1, u_2, \dots, u_n$ can be any functions of our desire, we can choose them in such a way that

$$u_1'y_1 + \dots + u_n'y_n = 0$$

Taking the derivative again,

$$y_p'' = (u_1y_1'' + \dots + u_ny_n'') + (u_1'y_1' + \dots + u_n'y_n')$$

and again, we impose the condition

$$u_1'y_1' + \dots + u_n'y_n' = 0.$$

We keep going on to compute $y_p''', \dots y_p^{(n-1)}$, and impose similar conditions; that is, in general, the conditions are

$$\begin{aligned} &u_1'y_1^{(0)} + \dots + u_n'y_n^{(0)} = 0 \\ &\vdots \\ &u_1'y_1^{(n-2)} + \dots + u_n'y_n^{(n-2)} = 0 \end{aligned}$$

In summary, for any $1 \le i \le n-1$,

$$y_p^{(i)} = \sum_{k=1}^n u_ky_k^{(i)} + \underbrace{\sum_{k=1}^n u_k'y_k^{(i-1)}}_{0} = \sum_{k=1}^n u_ky_k^{(i)}$$

For the final derivative, we have

$$\begin{aligned} y_p^{(n)} &= \left(u_1y_1^{(n)} + \dots + u_ny_n^{(n)}\right) + \left(u_1'y_1^{(n-1)} + \dots + u_n'y_n^{(n-1)}\right) \\ &= \sum_{k=1}^n u_ky_k^{(n)} + \sum_{k=1}^n u_k'y_k^{(n-1)} \end{aligned}$$

Replace $y_p, y_p', \dots y_p^{(n-1)}, y_p^{(n)}$ back into $Ly = F(x)$, we obtain

$$\begin{gathered} \left(\sum_{k=1}^n u_ky_k^{(n)} + \sum_{k=1}^n u_k'y_k^{(n-1)}\right) + a_{n-1}\sum_{k=1}^n u_ky_k^{(n-1)} + \dots \\ + a_1\sum_{k=1}^n u_ky_k' + a_0\sum_{k=1}^n u_ky_k = F(x) \end{gathered}$$

Rearranging a bit, we obtain

$$\begin{gathered} u_1\left(y_1^{(n)} + a_{n-1}y_1^{n-1} + \dots + a_1y_1' + a_0y_1\right) \\ + u_2\left(y_2^{(n)} + a_{n-1}y_2^{n-1} + \dots + a_1y_1' + a_0y_2\right) \\ \dots + u_n\left(y_n^{(n)} + a_{n-1}y_n^{n-1} + \dots + a_1y_n' + a_0y_n\right) \\ + \left(u_1'y_1^{(n-1)} + u_2'y_2^{(n-1)} + \dots + u_n'y_n^{(n-1)}\right) = F(x) \end{gathered}$$

As seen, the first n terms are $u_1Ly_1 + u_2Ly_2 + \dots + u_nLy_n$, which are all zeroes, since they are all solutions to $Ly = 0$. This left us with

$$u_1'y_1^{(n-1)} + u_2'y_2^{(n-1)} + \dots + u_n'y_n^{(n-1)} = F(x)$$

We now have a system of linear equations with n variables $u_1', u_2', \dots, u_n'$.

$$\begin{aligned} y_1u_1' + \dots + y_nu_n' &= 0 \\ y_1'u_1' + \dots + y_n'u_n' &= 0 \\ y_1^{(n-1)}u_1' + \dots + y_n^{(n-1)}u_n' &= F(x) \end{aligned}$$

The sufficient conditions for the uniqueness of the solution is the determinant of the matrix made up of the coefficients of the variables being different from zero. However, this determinant is just the Wronskian:

$$W(y_1, y_2, ..., y_n)(x) = W(x) = \begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix}$$

which is never zero on the interval I since $y_1, y_2, \ldots, y_n$ are linearly independent by hypothesis. Apply Crammer's Rule, we obtain

$$u_i' = \frac{W_i(x)F(x)}{W(x)}, \quad 1 \le i \le n$$

where $W_i(x)$ is the determinant of the matrix $W(x)$ obtained by replacing the i-th column of $W(x)$ by $\left[0, 0, \ldots, 1\right]^{\mathrm{T}}$. (Remember the property of determinant, we can pull out the constant here, which is $F(x)$).

Integrating u_i' gives

$$u_i(x) = \int \frac{W_i(x)F(x)}{W(x)} dx, \quad 1 \le i \le n,$$

and the final particular solution is

$$y_p = u_1 y_1 + u_2 y_2 + \ldots + u_n y_n.$$

The integration will leave a constant behind, but it doesn't matter at all since that constant will be part of $C_i y_i$ anyway.

20.2 Second Order Case

For the case of $n = 2$, we have

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}, \quad W_1 = \begin{vmatrix} 0 & y_2 \\ F(x) & y_2' \end{vmatrix}, \quad W_2 = \begin{vmatrix} y_1 & 0 \\ y_1' & F(x) \end{vmatrix},$$

and therefore we have

$$u_1' = \frac{-y_2 F(x)}{W(x)}, \quad \text{and} \quad u_2' = \frac{y_1 F(x)}{W(x)}$$

thus,

$$u_1 = \int \frac{-y_2 F(x)}{W(x)} dx, \quad \text{and} \quad u_2 = \int \frac{y_1 F(x)}{W(x)} dx$$

21 Cauchy-Euler Equation

Cauchy-Euler Equation. The Cauchy-Euler equation is a differential equation of the form

$$x^n y^{(n)} + a_{n-1} x^{n-1} y^{(n-1)} + \ldots + a_1 x y' + a_0 y = F(x).$$

To solve this equation, we must consider the homogeneous case where $F(x) = 0$, and then apply VoP to solve for the general solution.

We assume the trial solution

$$y = x^r.$$

By then

$$y' = r x^{r-1},\ y'' = r(r-1), \ldots,\ y^{(n)} = r(r+1)\ldots(r+n-1)x^{r-n}.$$

For convenience, we denote

$$p_n(r) = r(r+1)...(r+n-1), \quad y^{(n)} = p_n(r) x^{r-n}.$$

Substitute this back into the equation, and simplify x^r on both sides gives

$$p_n(r) + a_{n-1} p_{n-1}(r) + \ldots + a_1 p_1(r) + a_0 = 0,$$

which is a polynomial of degree n.

There are no simple ways to find the general coefficient of $p_n(r)$, but its coefficient is related to the unsigned Stirling number of the first kind, expressed as

$$p_n(r) = r(r+1)...(r+n-1) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} r^k$$

Second-Order Case. The second-order case reduces the characteristic polynomial to

$$r^2 + (a_1 - 1)r + a_0 = 0.$$

Again, there are three types of solutions:

1. **Case 1: Two Real Solutions.** If r_1 and r_2 are real solutions, then $\{x^{r_1}, x^{r_2}\}$ is a basis for S, and the solution is
$$y = C_1 x^{r_1} + C_2 x^{r_2}.$$
2. **Case 2: One Real Solution.** If r is a real solution, then $\{x^r, x^r \ln x\}$ is a basis for S, and the solution to the DE is
$$y = C_1 x^r + C_2 x^r \ln x.$$
3. **Case 3: Two Complex Solutions.** If $a \pm bi$ are complex solutions to $p(r)$, then $\{x^a \cos(b \ln x), x^a \sin(b \ln x)\}\}$ is a basis for S, and the solution to the DE is
$$y = x^a \left(C_1 \cos(b \ln x) + C_2 \sin(b \ln x)\right).$$

Non-homogeneous Equation. For the non-homogeneous case of

$$x^2 y'' + a_1 x y' + a_0 y = F(x),$$

first, express it in the form

$$y'' + \frac{1}{x} a_1 y' + \frac{1}{x^2} a_0 y = \frac{F(x)}{x^2}.$$

We know previously that y_1 and y_2 are solutions to the homogeneous equation, therefore $y_p = u_1 y_1 + u_2 y_2$. Apply the VoP formula, we obtain

$$u_1 = \int \frac{-y_2 F(x)}{x^2 W(x)} dx, \quad \text{and} \quad u_2 = \int \frac{y_1 F(x)}{x^2 W(x)} dx.$$

Therefore, the general solution is $y = C_1 y_1 + C_2 y_2 + u_1 y_1 + u_2 y_2$.

LAPLACE TRANSFORM AND THE GAMMA FUNCTION

22 Definitions

Laplace Transform. Let f be a continuous function defined on $[0, +\infty)$. We use t for the independent variable of f as a convention. The Laplace transform of f, written $\mathcal{L}\{f\}(s)$ is defined by

$$\mathcal{L}\{f\}(s) = \int_0^\infty f(t)e^{-st}dt = F(s),$$

where $s = \sigma + i\omega$ is a complex number, and $\sigma, \omega \in \mathbb{R}$.

It can be seen that $\mathcal{L}\{f\}(s)$ is a function of s, and we write $F(s)$ for $\mathcal{L}\{f\}(s)$. Clearly, $\mathcal{L}\{f\}(s)$ is a linear transformation, and therefore it satisfies

$$\mathcal{L}\{\alpha f + \beta g\}(s) = \alpha\mathcal{L}\{f\}(s) + \beta\mathcal{L}\{g\}(s)$$

Example. For $f(t) = t$,

$$\begin{aligned}\mathcal{L}\{t\}(s) &= \int_0^\infty te^{-st}dt = \lim_{k\to\infty}\int_0^k te^{-st}dt \\ &= \lim_{k\to\infty}\left(-\frac{t}{s}e^{-st} - \frac{1}{s^2}e^{-st}\Big|_0^k\right) \\ &= \lim_{k\to\infty}\left(-\frac{k}{s}e^{-ks} - \frac{1}{s^2}e^{-ks} + 0 + \frac{1}{s^2}\right) \\ &= \frac{1}{s^2}, \quad \text{for } s > 0\end{aligned}$$

Exponential Order. A function $f(t)$ is said to be of exponential order α if there exists positive constants T and M such that

$$|f(t)| \le Me^{\alpha t}, \quad \forall t \ge T.$$

Existence of the Laplace. If $f(t)$ is piecewise continuous on $[0, \infty)$ and is of exponential order α, then $\mathcal{L}\{f\}(s)$ exists for all $s > \alpha$. This is true due to the comparison test.

For example, $f(t) = e^{t^2/2}$ is clearly not of exponential order, and therefore $\mathcal{L}\left\{e^{t^2/2}\right\}(s)$ does not exist.

Properties of Laplace Transform. Given that $\mathcal{L}\{f(t)\} = F(s)$.

1. Linearity: $\mathcal{L}\{\alpha f + \beta g\} = \alpha\mathcal{L}\{f\} + \beta\mathcal{L}\{g\}$.
2. $\mathcal{L}\{f'\} = sF(s) - f(0)$.
3. $\mathcal{L}\{f''\} = s^2F(s) - sf(0) - f'(0)$.
4. $\mathcal{L}\left\{f^{(n)}\right\} = s^nF(s) - s^{n-1}f(0) - s^{n-2}f'(0) - \ldots - f^{(n-1)}(0)$.
5. $\mathcal{L}\{t^nf(t)\} = (-1)^nF^{(n)}(s)$
6. First shifting: $\mathcal{L}\left\{e^{at}f(t)\right\} = F(s-a)$.
7. Second shifting: $\mathcal{L}\{u_a(t)f(t-a)\} = e^{-as}F(s)$, and $\mathcal{L}\{u_a(t)f(t)\} = e^{-as}\mathcal{L}\{f(t+a)\}$.

Inverse Laplace Transform. Assume that f is a piecewise continuous function on $[0, +\infty)$, and f is of exponential order. In this case, $\mathcal{L}\{f\}(s)$ is one-to-one, and $\mathcal{L}^{-1}\{F(s)\}(t)$ exits.

Since $\mathcal{L}$ is a LT, $\mathcal{L}^{-1}$ must also be a LT.

Example. Find the inverse Laplace of

$$F(s) = \frac{2s+3}{(s-2)(s^2+1)}.$$

To solve this, see the table of Laplace transform below. Using partial fraction decomposition, we see that

$$F(s) = \frac{7}{5}\frac{1}{s-2} - \frac{1}{5}\frac{7s+4}{s^2+1} = \frac{7}{5}\frac{1}{s-2} - \frac{7}{5}\frac{s}{s^2+1} - \frac{4}{5}\frac{1}{s^2+1}$$

and therefore

$$\begin{aligned}\mathcal{L}^{-1}\{F(s)\} &= \frac{7}{5}\mathcal{L}^{-1}\left\{\frac{1}{s-2}\right\} - \frac{7}{5}\mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\} - \frac{1}{5}\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} \\ &= \frac{7}{5}e^{2t} - \frac{7}{5}\cos(t) - \frac{1}{5}\sin(t)\end{aligned}$$

The Unit Step Function. Let a be a positive real number. The unit step function $u_a(t)$, sometimes called the Heaviside function is defined by

$$u_a(t) = u_0(t-a) = \begin{cases}0 & \text{if } 0 \le t < a \\ 1 & \text{if } t \ge a\end{cases}$$

Some piecewise functions can be expressed as a combination of other functions and the unit step function. For example,

$$f(t) = \begin{cases}t & \text{if } 0 \le t < 3 \\ 6-t & \text{if } 3 \le t < 6 \\ 0 & \text{if } t \ge 6\end{cases}$$

Then,

$$\begin{aligned}f(t) &= t[u_0(t) - u_3(t)] + (6-t)[u_3(t) - u_6(t)] \\ &= t - (2t-6)u_3(t) + (t-6)u_6(t)\end{aligned}$$

Another example is

$$f(t) = \begin{cases}\sin t, & 2k\pi \le t \le (2k+1)\pi,\ k \in \mathbb{Z}^+ \\ 0, & \text{otherwise}\end{cases}$$

Then,

$$\begin{aligned}f(t) &= \sin t[u_{0\pi}(t) - u_\pi(t)] + \sin t[u_{2\pi}(t) - u_{3\pi}(t)] + \ldots \\ &= \sin t\sum_{k=1}^\infty\left(u_{2k\pi}(t) - u_{(2k+1)\pi}(t)\right)\end{aligned}$$

Table of Laplace Transform. This is just a basic table, there are way more functions in practice.

$f(t) = \mathcal{L}^{-1}\{F\}$	$F(s) = \mathcal{L}\{f\}$	$D(\mathcal{L})$
1	$\frac{1}{s}$	$\text{Re}(s) > 0$
e^{at}	$\frac{1}{s-a}$	$\text{Re}(s) > a$
$t^n,\ n \in \mathbb{Z}^+$	$\frac{n!}{s^{n+1}}$	$\text{Re}(s) > 0$
$\sin bt$	$\frac{b}{s^2+b^2}$	$\text{Re}(s) > 0$
$\cos bt$	$\frac{s}{s^2+b^2}$	$\text{Re}(s) > 0$
$\delta(t-a)$	e^{-as}	$\text{Re}(s) > 0$
$u_a(t) = u_0(t-a)$	$\frac{e^{-as}}{s}$	$\text{Re}(s) > 0$
$e^{at}f(t)$	$F(s-a)$	$\text{Re}(s) > 0$
$u_a(t)f(t-a)$	$e^{-as}F(s)$	$\text{Re}(s) > 0$

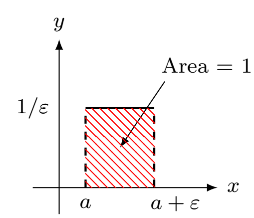
23 The Dirac Delta "Function"

Background. There are many ways to define this function, one way is to start by defining

$$d_\varepsilon(t-a) = \frac{u_a(t) - u_{a+\varepsilon}(t)}{\varepsilon};$$

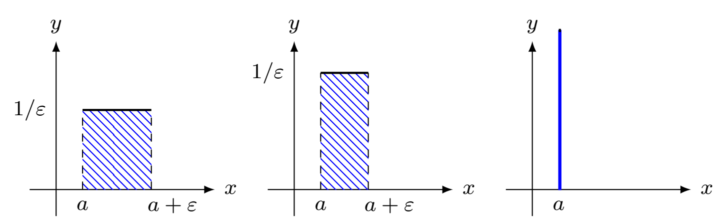
that is, as illustrated below:

$$\begin{aligned}A &= \int_0^\infty d_\varepsilon(t-a)dt \\ &= \int_a^{a+\varepsilon}\frac{1}{\varepsilon}dt = 1\end{aligned}$$



The Dirac delta function is defined as

$$\delta(t-a) = \lim_{\varepsilon\to 0^+} d_\varepsilon(t-a).$$



As we can see, the box slowly becomes thinner, and taller. As $\varepsilon \to 0^+$, $1/\varepsilon \to \infty$, and the box becomes infinitely tall and infinitely thin, but the area under the line is still one.

The Dirac Delta Function. The Dirac delta function at a, written $\delta(t-a)$ is defined as

$$\delta(t-a) = \begin{cases}0 & \text{if } t \ne a \\ \infty & \text{if } t = a\end{cases}$$

Another property that the Dirac delta satisfies is

$$\int_0^\infty \delta(t-a)dt = 1.$$

The Laplace transform of the Dirac delta is

$$\mathcal{L}\left\{\delta(t-a)\right\}(s) = e^{-as}.$$

Note. The Dirac Delta is NOT really a function by any means.

Physical Application.

- The Laplace transform in mechanics problems can be understood as an impulse force, whose magnitude is very large, but happens through a very brief period of time.
- In electricity, the Laplace transform can also be used as a switch.

24 Solving DE using Laplace Transform

Strategy. Given an IVP

$$a_n y^{(n)} + a_{n-1}y^{(n-1)} + \ldots + a_1 y' + a_0 y = F(t), \quad y(0) = y_0,\ y'(0) = y_1,\ \ldots$$

1. Apply $\mathcal{L}$ to both sides of the DE.
2. Using the properties of the Laplace, substitute the initial conditions, and solve for $Y(s) = \mathcal{L}\left\{y(t)\right\}$.
3. Apply $\mathcal{L}^{-1}$ to $Y(s)$ to recover $y(t) = \mathcal{L}^{-1}\left\{Y(s)\right\}$.

Example. Solve the IVP

$$y'' - 3y' + 2y = e^{3t}, \quad y(0) = 1,\ y'(0) = 0.$$

Solution. Apply $\mathcal{L}$ to both sides:

$$\mathcal{L}\left\{y''\right\} - 3\mathcal{L}\left\{y'\right\} + 2\mathcal{L}\left\{y\right\} = \mathcal{L}\left\{e^{3t}\right\}$$

$$[s^2Y(s) - sy(0) - y'(0)] - 3[sY(s) - y(0)] + 2Y(s) = \frac{1}{s-3}$$

Substituting the initial conditions $y(0) = 1$, $y'(0) = 0$:

$$(s^2 - 3s + 2)Y(s) = s - 3 + \frac{1}{s-3}$$

$$Y(s) = \frac{s-3}{(s-1)(s-2)} + \frac{1}{(s-1)(s-2)(s-3)}$$

Using partial fractions:

$$Y(s) = \frac{5/2}{s-1} - \frac{2}{s-2} + \frac{1/2}{s-3}$$

Therefore,

$$y(t) = \frac{5}{2}e^t - 2e^{2t} + \frac{1}{2}e^{3t}$$

Example. Solve the IVP

$$y'' + y = \delta(t-\pi), \quad y(0) = 0,\ y'(0) = 1.$$

Solution. Apply $\mathcal{L}$ to both sides:

$$s^2Y(s) - sy(0) - y'(0) + Y(s) = e^{-\pi s}$$
$$(s^2+1)Y(s) = 1 + e^{-\pi s}$$
$$Y(s) = \frac{1}{s^2+1} + \frac{e^{-\pi s}}{s^2+1}$$

Taking the inverse Laplace:

$$y(t) = \sin t + u_\pi(t)\sin(t-\pi) = \sin t - u_\pi(t)\sin t$$
$$= \begin{cases} \sin t, & 0 \le t < \pi \\ 0, & t \ge \pi \end{cases}$$

25 Convolution

Definition. The convolution of two functions f and g, written $f * g$, is defined by

$$(f*g)(t) = \int_0^t f(\tau)g(t-\tau)\, d\tau.$$

Properties.

1. Commutativity: $f * g = g * f$.
2. Associativity: $f * (g * h) = (f * g) * h$.
3. Distributivity: $f * (g + h) = f * g + f * h$.
4. Identity: $f * \delta = f$.
5. Zero: $f * 0 = 0$.

Theorem 10 (**Convolution Theorem**)**.** If $\mathcal{L}\left\{f(t)\right\} = F(s)$ and $\mathcal{L}\left\{g(t)\right\} = G(s)$, then

$$\mathcal{L}\left\{(f*g)(t)\right\} = F(s)\cdot G(s).$$

Equivalently,

$$\mathcal{L}^{-1}\left\{F(s)\cdot G(s)\right\} = \int_0^t f(\tau)g(t-\tau)\, d\tau.$$

Example. Find the inverse Laplace of

$$H(s) = \frac{1}{(s-1)(s^2+1)}.$$

Solution. Let $F(s) = \frac{1}{s-1}$ and $G(s) = \frac{1}{s^2+1}$, so $f(t) = e^t$ and $g(t) = \sin t$. By the convolution theorem:

$$\mathcal{L}^{-1}\left\{H(s)\right\} = (f*g)(t) = \int_0^t e^\tau \sin(t-\tau)\, d\tau = \frac{1}{2}\left(e^t - \cos t - \sin t\right)$$

26 The Gamma Function

Definition. The Gamma function, $\Gamma(x)$, is defined for $x > 0$ by the improper integral

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}\, dt.$$

Properties.

1. Recursive property: $\Gamma(x+1) = x\Gamma(x)$, for $x > 0$.
2. $\Gamma(1) = 1$, and for positive integers, $\Gamma(n+1) = n!$.
3. $\Gamma(1/2) = \sqrt{\pi}$.
4. Reflection formula: $\Gamma(x)\Gamma(1-x) = \dfrac{\pi}{\sin(\pi x)}$, for $x \notin \mathbb{Z}$.
5. Duplication formula: $\Gamma(x)\Gamma\left(x + \frac{1}{2}\right) = \dfrac{\sqrt{\pi}}{2^{2x-1}}\Gamma(2x)$.

Laplace Transform of t^α. For $\alpha > -1$,

$$\mathcal{L}\left\{t^\alpha\right\} = \frac{\Gamma(\alpha+1)}{s^{\alpha+1}}, \quad s > 0.$$

When $\alpha = n \in \mathbb{Z}^+$, this reduces to $\mathcal{L}\left\{t^n\right\} = \frac{n!}{s^{n+1}}$, consistent with the table above.

Example. Find $\mathcal{L}\left\{t^{1/2}\right\}$.
Solution.

$$\mathcal{L}\left\{t^{1/2}\right\} = \frac{\Gamma(3/2)}{s^{3/2}} = \frac{\frac{1}{2}\Gamma(1/2)}{s^{3/2}} = \frac{\sqrt{\pi}}{2s^{3/2}}$$

SYSTEM OF DIFFERENTIAL EQUATIONS

27 General Theory

System of First-Order DEs. A system of first-order linear DEs is a set of n equations

$$\mathbf{x}'(t) = A(t)\mathbf{x}(t) + \mathbf{f}(t),$$

where $\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{bmatrix}$, $A(t)$ is an $n \times n$ matrix, and $\mathbf{f}(t)$ is a forcing vector. When $\mathbf{f}(t) = \mathbf{0}$, the system is homogeneous: $\mathbf{x}' = A\mathbf{x}$.

Conversion. Any n-th order linear DE can be converted to a system of n first-order DEs. Given

$$y^{(n)} + a_{n-1}y^{(n-1)} + \ldots + a_1y' + a_0y = F(t),$$

let $x_1 = y$, $x_2 = y'$, $\ldots$, $x_n = y^{(n-1)}$. Then,

$$\begin{bmatrix} x_1' \\ x_2' \\ \vdots \\ x_{n-1}' \\ x_n' \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-1} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ F(t) \end{bmatrix}$$

Theorem 11 (**Existence and Uniqueness**)**.** If $A(t)$ and $\mathbf{f}(t)$ are continuous on an open interval I containing t_0, then the IVP $\mathbf{x}' = A(t)\mathbf{x} + \mathbf{f}(t)$, $\mathbf{x}(t_0) = \mathbf{x}_0$ has a unique solution on I.

Fundamental Matrix. If $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$ are n linearly independent solutions to $\mathbf{x}' = A\mathbf{x}$, then the matrix

$$\Phi(t) = \begin{bmatrix} \mathbf{x}_1 & \mathbf{x}_2 & \cdots & \mathbf{x}_n \end{bmatrix}$$

is called a fundamental matrix of the system. The general solution is $\mathbf{x} = \Phi(t)\mathbf{c}$, where $\mathbf{c}$ is a constant vector.

Wronskian of Systems. The Wronskian of the system is $W(t) = \det\Phi(t)$. If $W(t_0) \neq 0$ for some $t_0 \in I$, then $W(t) \neq 0$ for all $t \in I$ (Abel's formula):

$$W(t) = W(t_0)\exp\left(\int_{t_0}^t \operatorname{tr}(A(\tau))\, d\tau\right).$$

28 Constant Coefficient Homogeneous Systems

Strategy. For $\mathbf{x}' = A\mathbf{x}$ where A is a constant $n \times n$ matrix, try $\mathbf{x} = \mathbf{v}e^{\lambda t}$. Then $\lambda \mathbf{v}e^{\lambda t} = A\mathbf{v}e^{\lambda t}$, so $A\mathbf{v} = \lambda\mathbf{v}$. This is the eigenvalue problem for A.

Case 1: Distinct Real Eigenvalues. If A has n distinct real eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ with eigenvectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$, then the general solution is

$$\mathbf{x}(t) = C_1\mathbf{v}_1e^{\lambda_1 t} + C_2\mathbf{v}_2e^{\lambda_2 t} + \ldots + C_n\mathbf{v}_ne^{\lambda_n t}.$$

Case 2: Complex Eigenvalues. If $\lambda = \alpha + \beta i$ is an eigenvalue with eigenvector $\mathbf{v} = \mathbf{a} + i\mathbf{b}$, then $\bar{\lambda} = \alpha - \beta i$ is also an eigenvalue with eigenvector $\bar{\mathbf{v}} = \mathbf{a} - i\mathbf{b}$. Two real-valued solutions are:

$$\mathbf{x}_1 = e^{\alpha t}(\mathbf{a}\cos\beta t - \mathbf{b}\sin\beta t), \quad \mathbf{x}_2 = e^{\alpha t}(\mathbf{a}\sin\beta t + \mathbf{b}\cos\beta t).$$

Case 3: Repeated Eigenvalues. If λ is an eigenvalue of algebraic multiplicity m but geometric multiplicity $k < m$, we need generalized eigenvectors. For a 2×2 system with λ repeated and only one eigenvector $\mathbf{v}$, find a generalized eigenvector $\mathbf{w}$ satisfying $(A - \lambda I)\mathbf{w} = \mathbf{v}$. Then,

$$\mathbf{x}(t) = C_1\mathbf{v}e^{\lambda t} + C_2(\mathbf{v}t + \mathbf{w})e^{\lambda t}.$$

Example. Solve $\mathbf{x}' = \begin{bmatrix} 1 & -2 \\ 3 & -4 \end{bmatrix}\mathbf{x}$.

Solution. The characteristic polynomial is

$$\det(A - \lambda I) = \lambda^2 + 3\lambda + 2 = (\lambda+1)(\lambda+2) = 0$$

so $\lambda_1 = -1$ and $\lambda_2 = -2$. For $\lambda_1 = -1$: $(A + I)\mathbf{v} = \mathbf{0}$ gives $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. For $\lambda_2 = -2$: $(A + 2I)\mathbf{v} = \mathbf{0}$ gives $\mathbf{v}_2 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$. The general solution is

$$\mathbf{x}(t) = C_1\begin{bmatrix} 1 \\ 1 \end{bmatrix}e^{-t} + C_2\begin{bmatrix} 2 \\ 3 \end{bmatrix}e^{-2t}$$

29 The Matrix Exponential

Definition. The matrix exponential of an $n \times n$ matrix A is defined by

$$e^{At} = \sum_{k=0}^{\infty}\frac{(At)^k}{k!} = I + At + \frac{A^2t^2}{2!} + \frac{A^3t^3}{3!} + \ldots$$

The solution to $\mathbf{x}' = A\mathbf{x}$, $\mathbf{x}(0) = \mathbf{x}_0$ is $\mathbf{x}(t) = e^{At}\mathbf{x}_0$.

Properties.

1. $e^O = I$ (where O is the zero matrix).
2. $\dfrac{d}{dt}e^{At} = Ae^{At} = e^{At}A$.
3. If $AB = BA$, then $e^{(A+B)t} = e^{At}e^{Bt}$.
4. $(e^{At})^{-1} = e^{-At}$.
5. If $A = PDP^{-1}$ (diagonalizable), then $e^{At} = Pe^{Dt}P^{-1}$, where $e^{Dt} = \mathrm{diag}(e^{\lambda_1 t}, \ldots, e^{\lambda_n t})$.

30 Non-homogeneous Systems

Variation of Parameters for Systems. The particular solution of $\mathbf{x}' = A\mathbf{x} + \mathbf{f}(t)$ is

$$\mathbf{x}_p(t) = \Phi(t)\int \Phi^{-1}(t)\mathbf{f}(t)\,dt,$$

where $\Phi(t)$ is a fundamental matrix. With the matrix exponential:

$$\mathbf{x}(t) = e^{At}\mathbf{x}_0 + \int_0^t e^{A(t-\tau)}\mathbf{f}(\tau)\,d\tau.$$

31 Phase Portraits (2×2 Systems)

For a 2×2 system $\mathbf{x}' = A\mathbf{x}$ with eigenvalues λ_1, λ_2:

Eigenvalues	Type	Stability
$\lambda_1 < \lambda_2 < 0$	Stable node	Asympt. stable
$0 < \lambda_1 < \lambda_2$	Unstable node	Unstable
$\lambda_1 < 0 < \lambda_2$	Saddle point	Unstable
$\alpha \pm \beta i$, $\alpha < 0$	Stable spiral	Asympt. stable
$\alpha \pm \beta i$, $\alpha > 0$	Unstable spiral	Unstable
$\pm\beta i$	Center	Stable (not asympt.)
$\lambda_1 = \lambda_2 < 0$ (2 e.v.)	Stable star	Asympt. stable
$\lambda_1 = \lambda_2 < 0$ (1 e.v.)	Degenerate stable	Asympt. stable

32 Nonlinear Systems and Stability

Autonomous Systems. A system $\mathbf{x}' = \mathbf{F}(\mathbf{x})$ (no explicit dependence on t) is called autonomous. An equilibrium (critical) point is a point $\mathbf{x}_0$ where $\mathbf{F}(\mathbf{x}_0) = \mathbf{0}$.

Stability Definitions. Let $\mathbf{x}_0$ be an equilibrium point of $\mathbf{x}' = \mathbf{F}(\mathbf{x})$.

- $\mathbf{x}_0$ is **stable** if for every $\varepsilon > 0$, there exists $\delta > 0$ such that $\|\mathbf{x}(0) - \mathbf{x}_0\| < \delta$ implies $\|\mathbf{x}(t) - \mathbf{x}_0\| < \varepsilon$ for all $t > 0$.
- $\mathbf{x}_0$ is **asymptotically stable** if it is stable and $\mathbf{x}(t) \to \mathbf{x}_0$ as $t \to \infty$.
- $\mathbf{x}_0$ is **unstable** if it is not stable.

Linearization. Given a nonlinear system

$$\mathbf{x}' = \mathbf{F}(\mathbf{x}), \quad \text{equilibrium at } \mathbf{x}_0,$$

let $\mathbf{u} = \mathbf{x} - \mathbf{x}_0$. The linearized system near $\mathbf{x}_0$ is

$$\mathbf{u}' \approx J(\mathbf{x}_0)\mathbf{u},$$

where $J(\mathbf{x}_0)$ is the Jacobian matrix evaluated at $\mathbf{x}_0$:

$$J = \begin{bmatrix} \dfrac{\partial F_1}{\partial x_1} & \dfrac{\partial F_1}{\partial x_2} \\ \dfrac{\partial F_2}{\partial x_1} & \dfrac{\partial F_2}{\partial x_2} \end{bmatrix}_{\mathbf{x}=\mathbf{x}_0}$$

Theorem 12 (**Hartman-Grobman**). If the eigenvalues of $J(\mathbf{x}_0)$ all have nonzero real parts (i.e., $\mathbf{x}_0$ is a hyperbolic equilibrium), then the nonlinear system behaves qualitatively like the linearized system near $\mathbf{x}_0$. In particular:

- If all eigenvalues have $\mathrm{Re}(\lambda) < 0$, then $\mathbf{x}_0$ is asymptotically stable.
- If any eigenvalue has $\mathrm{Re}(\lambda) > 0$, then $\mathbf{x}_0$ is unstable.

If $\mathrm{Re}(\lambda) = 0$ for some eigenvalue (e.g., a center), linearization is inconclusive and higher-order analysis or Lyapunov methods are needed.

Example. Classify the equilibria of

$$x' = x - xy, \quad y' = -y + xy.$$

Solution. Setting $x' = y' = 0$: $x(1-y) = 0$ and $y(-1+x) = 0$. The equilibria are $(0,0)$ and $(1,1)$. The Jacobian is

$$J = \begin{bmatrix} 1-y & -x \\ y & -1+x \end{bmatrix}.$$

At $(0,0)$: $J = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, eigenvalues $\lambda = 1, -1 \Rightarrow$ saddle (unstable).
At $(1,1)$: $J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, eigenvalues $\lambda = \pm i \Rightarrow$ center for the linear system.
Since the eigenvalues are purely imaginary, linearization is inconclusive. (This is a Lotka-Volterra system; the equilibrium is in fact a center.)

Lyapunov's Direct Method. Instead of solving the system, we can determine stability by finding a Lyapunov function $V(\mathbf{x})$ satisfying:

1. $V(\mathbf{x}_0) = 0$ and $V(\mathbf{x}) > 0$ for $\mathbf{x} \neq \mathbf{x}_0$ (positive definite).
2. $\dot{V} = \nabla V \cdot \mathbf{F}(\mathbf{x}) \leq 0$ along trajectories.

If $\dot{V} < 0$ (negative definite), then $\mathbf{x}_0$ is asymptotically stable. If $\dot{V} \leq 0$ (negative semi-definite), then $\mathbf{x}_0$ is stable.

Example. Show that the origin is asymptotically stable for

$$x' = -x + y^2, \quad y' = -x^2 - y^3.$$

Solution. Try $V(x,y) = x^2 + y^2$ (positive definite). Then

$$\dot{V} = 2x(-x + y^2) + 2y(-x^2 - y^3) = -2x^2 + 2xy^2 - 2x^2y - 2y^4.$$

$$= -2x^2 - 2y^4 + 2xy(y - x) = -2(x^2 + y^4) + 2xy(y - x).$$

Near the origin, $-2(x^2 + y^4)$ dominates, so $\dot{V} < 0$ in a neighborhood of the origin. Therefore, the origin is asymptotically stable.

Nullclines. For the $2D$ system $x' = F(x,y)$, $y' = G(x,y)$:

- The x-nullcline is the set where $F(x,y) = 0$ (trajectories are vertical on this curve).
- The y-nullcline is the set where $G(x,y) = 0$ (trajectories are horizontal on this curve).
- Equilibria occur at the intersection of nullclines.

Nullclines divide the phase plane into regions where the signs of x' and y' are constant, which helps sketch the phase portrait.

SERIES METHOD, NON-ELEMENTARY SOLUTION

33 General Theory and Tools

Power Series.

1. **Definition.** A power series centered around a is an infinite series of the form
$$\sum_{n=0}^{\infty} c_n(x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \dots$$
2. **Interval of Convergence.** Every power series has an interval of convergence, which is the set of all real, or complex numbers x for which the series converges.
3. **Radius of Convergence.** If $\lim_{n\to\infty}\left|\frac{c_{n+1}}{c_n}\right| = \alpha$, or if $\lim_{n\to\infty}\sqrt[n]{|c_n|} = \alpha$, then the radius of convergence of the power series
$$\sum_{n=0}^{\infty} c_n(x-a)^n$$
is determined by
$$R = \begin{cases} 1/\alpha, & \text{if } 0 < \alpha < \infty \\ 0, & \text{if } \alpha = +\infty \\ +\infty, & \text{if } \alpha = 0 \end{cases}$$
and the interval of convergence is $(a-R, a+R)$.
4. **Absolute Convergence.** If $x \in (a-R, a+R)$, and x is not at the endpoint of the interval, then the series
$$\sum_{n=0}^{\infty} |c_n(x-a)^n|$$
converges. In other words, any $x \in (a-R, a+R)$ makes the power series converge absolutely.
5. **Vanishing Power Series.** If $\sum_{n=0}^{\infty} c_n(x-a)^n = 0$, $\forall x \in I \subseteq \mathbb{R}$, then $c_n = 0$, for each $n = 0, 1, 2, \dots$.
6. **Differentiation.** Given a power series $f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n$ with radius of convergence $R > 0$, then f is differentiable on the interval $|x-a| < R$ and,
$$f'(x) = \sum_{n=1}^{\infty} nc_n(x-a)^{n-1},$$
$$f''(x) = \sum_{n=2}^{\infty} n(n-1)c_n(x-a)^{n-2},$$
and in general,
$$f^{(k)}(x) = \sum_{n=k}^{\infty} n(n-1)\dots(n-k+1)c_n(x-a)^{n-k}.$$
All of the series still have the same radius of convergence of R.
7. **Integration.** For that same power series and radius of convergence,
$$\int f(x)\,dx = \int\left(\sum_{n=0}^{\infty} c_n(x-a)^n\right)dx = \sum_{n=0}^{\infty}\frac{c_n}{n+1}(x-a)^{n+1} + C,$$
and,
$$\int\left(\sum_{n=0}^{\infty} c_n(x-a)^n\right)dx = \sum_{n=1}^{\infty}\int_0^t c_n(x-a)^n\,dx, \quad \forall t \in (a-R, a+R)$$
8. **Index Shifting.** Given a finite summation $\sum_{i=n_0}^{N} f(i)$. Then, we can shift the index of the summation by any factor of k without changing the summation value.
$$\sum_{i=n_0}^{N} a_i = \sum_{i=n_0-k}^{N+k} a_{i+k}.$$
For an infinite series,
$$\sum_{i=n_0}^{\infty} a_i = \sum_{i=n_0-k}^{\infty} a_{i+k}.$$

Analytic Function. A function $f(x)$ is said to be analytic at $x = x_0$ if it can be represented by a power series
$$f(x) = \sum_{n=0}^{\infty} c_n(x-x_0)^n$$
for $|x-x_0| < R$ with $R > 0$. All polynomials, e^x, $\sin x$, $\cos x$ are analytic everywhere. Rational functions are analytic except at their poles.

Ordinary Point and Singular Point. Consider the DE
$$P(x)y'' + Q(x)y' + R(x)y = 0.$$
A point x_0 is an **ordinary point** if $Q(x)/P(x)$ and $R(x)/P(x)$ are both analytic at x_0. If x_0 is not an ordinary point, it is called a **singular point**.

Regular Singular Point. Let
$$P(x)y'' + Q(x)y' + R(x)y = 0$$
be a linear second-order DE. Then, x_0 is called a regular singular point if
$$\lim_{x\to x_0}\frac{Q(x)}{P(x)}(x-x_0), \quad \text{and} \quad \lim_{x\to x_0}\frac{R(x)}{P(x)}(x-x_0)^2$$
are both finite. If either limit is not finite, x_0 is called an **irregular singular point**.

Example. For the Bessel equation $x^2y'' + xy' + (x^2-\alpha^2)y = 0$, rewrite as
$$y'' + \frac{1}{x}y' + \frac{x^2-\alpha^2}{x^2}y = 0.$$
At $x_0 = 0$: $xp(x) = x\cdot\frac{1}{x} = 1$ and $x^2q(x) = x^2\cdot\frac{x^2-\alpha^2}{x^2} = x^2-\alpha^2$. Both limits are finite, so $x_0 = 0$ is a regular singular point.

34 Solving Differential Equations Using Series

Strategy. Given a DE with an ordinary point at x_0, assume a power series solution
$$y = \sum_{n=0}^{\infty} c_n(x-x_0)^n.$$

1. Compute y' and y'' (or higher derivatives as needed) as power series.
2. Substitute into the DE.
3. Shift indices so that all summations have the same power of $(x-x_0)$.
4. Set all coefficients of like powers to zero to obtain a recurrence relation for c_n.
5. Solve the recurrence to find the coefficients.

Theorem 13. If x_0 is an ordinary point of $y'' + p(x)y' + q(x)y = 0$, then the general solution is
$$y = \sum_{n=0}^{\infty} c_n(x-x_0)^n,$$
with radius of convergence at least as large as the distance from x_0 to the nearest singular point (in $\mathbb{C}$).

Example. Solve the DE $y'' - xy = 0$ near $x_0 = 0$.
Solution. Let $y = \sum_{n=0}^{\infty} c_nx^n$. Then $y'' = \sum_{n=2}^{\infty} n(n-1)c_nx^{n-2}$. Substituting:
$$\sum_{n=2}^{\infty} n(n-1)c_nx^{n-2} - x\sum_{n=0}^{\infty} c_nx^n = 0$$
$$\sum_{n=2}^{\infty} n(n-1)c_nx^{n-2} - \sum_{n=0}^{\infty} c_nx^{n+1} = 0$$
Shift $n \to n+2$ in the first sum and $n \to n-1$ in the second:
$$\sum_{n=0}^{\infty}(n+2)(n+1)c_{n+2}x^n - \sum_{n=1}^{\infty} c_{n-1}x^n = 0$$
$$2c_2 + \sum_{n=1}^{\infty}\left[(n+2)(n+1)c_{n+2} - c_{n-1}\right]x^n = 0$$
This gives $c_2 = 0$ and the recurrence relation
$$c_{n+2} = \frac{c_{n-1}}{(n+2)(n+1)}, \quad n \geq 1.$$
Starting with c_0 and c_1 arbitrary:
$$c_2 = 0, \quad c_3 = \frac{c_0}{3\cdot 2}, \quad c_4 = \frac{c_1}{4\cdot 3}, \quad c_5 = \frac{c_2}{5\cdot 4} = 0,$$
$$c_6 = \frac{c_3}{6\cdot 5} = \frac{c_0}{6\cdot 5\cdot 3\cdot 2}, \quad \dots$$
The solution is $y = c_0y_1(x) + c_1y_2(x)$, where y_1 and y_2 are the two linearly independent Airy functions.

Example. Solve $(1+x^2)y'' + 2xy' - 2y = 0$ near $x_0 = 0$.

Solution. The singular points are at $x = \pm i$, so $R \geq 1$. Let $y = \sum_{n=0}^{\infty} c_n x^n$. Substituting:

$$(1+x^2)\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} + 2x\sum_{n=1}^{\infty} nc_n x^{n-1} - 2\sum_{n=0}^{\infty} c_n x^n = 0$$

Expanding and collecting like powers:

$$\sum_{n=0}^{\infty}(n+2)(n+1)c_{n+2}x^n + \sum_{n=2}^{\infty} n(n-1)c_n x^n + \sum_{n=1}^{\infty} 2nc_n x^n - \sum_{n=0}^{\infty} 2c_n x^n = 0$$

For $n = 0$: $2c_2 - 2c_0 = 0$, so $c_2 = c_0$. For $n \geq 1$, the recurrence is

$$c_{n+2} = \frac{(2-n)(n+1)}{(n+2)(n+1)}c_n = \frac{2-n}{n+2}c_n.$$

For $n = 1$: $c_3 = \frac{1}{3}c_1$. For $n = 2$: $c_4 = 0$. All subsequent even terms vanish. The solution is

$$y = c_0(1+x^2) + c_1\left(x + \frac{x^3}{3} + \dots\right)$$

35 Frobenius Method

Strategy. The Frobenius method is used when x_0 is a regular singular point of the DE

$$P(x)y'' + Q(x)y' + R(x)y = 0.$$

Divide by $P(x)$ and write the DE in the standard form

$$y'' + p(x)y' + q(x)y = 0,$$

where $xp(x)$ and $x^2q(x)$ are analytic at $x_0 = 0$. Let $p_0 = \lim_{x\to 0} xp(x)$ and $q_0 = \lim_{x\to 0} x^2 q(x)$.

Indicial Equation. Assume a solution of the form

$$y = \sum_{n=0}^{\infty} c_n x^{n+r}, \quad c_0 \neq 0.$$

Substituting into the DE, the lowest power of x yields the indicial equation:

$$r(r-1) + p_0 r + q_0 = 0.$$

Let r_1 and r_2 be the roots of the indicial equation, with $\mathrm{Re}(r_1) \geq \mathrm{Re}(r_2)$.

Three Cases.

1. **Case 1:** $r_1 - r_2 \notin \mathbb{Z}$**.** Two linearly independent Frobenius solutions exist:
$$y_1 = \sum_{n=0}^{\infty} a_n x^{n+r_1}, \quad y_2 = \sum_{n=0}^{\infty} b_n x^{n+r_2}.$$
2. **Case 2:** $r_1 = r_2 = r$**.** One Frobenius solution $y_1 = \sum_{n=0}^{\infty} a_n x^{n+r}$. The second solution has the form
$$y_2 = y_1 \ln x + \sum_{n=0}^{\infty} b_n x^{n+r}.$$
3. **Case 3:** $r_1 - r_2 \in \mathbb{Z}^+$**.** One Frobenius solution $y_1 = \sum_{n=0}^{\infty} a_n x^{n+r_1}$. The second solution is
$$y_2 = Cy_1 \ln x + \sum_{n=0}^{\infty} b_n x^{n+r_2},$$
where the constant C may or may not be zero (determined by the recurrence).

Example. Solve $2x^2y'' + 3xy' - (1+x)y = 0$ near $x_0 = 0$.
Solution. Divide by $2x^2$: $y'' + \frac{3}{2x}y' - \frac{1+x}{2x^2}y = 0$. Here $p_0 = \lim_{x\to 0} x \cdot \frac{3}{2x} = \frac{3}{2}$ and $q_0 = \lim_{x\to 0} x^2 \cdot \left(-\frac{1+x}{2x^2}\right) = -\frac{1}{2}$. The indicial equation is

$$r(r-1) + \frac{3}{2}r - \frac{1}{2} = 0 \implies 2r^2 + r - 1 = 0 \implies (2r-1)(r+1) = 0$$

So $r_1 = 1/2$ and $r_2 = -1$. Since $r_1 - r_2 = 3/2 \notin \mathbb{Z}$, this is Case 1 and both Frobenius series converge.

SPECIAL FUNCTIONS AS SOLUTIONS TO DIFFERENTIAL EQUATIONS

36 Legendre Differential Equation

The equation of the form

$$(1-x^2)y'' - 2xy' + n(n+1)y = 0$$

where n is some parameter, is called Legendre's equation. In standard form:

$$y'' - \frac{2x}{1-x^2}y' + \frac{n(n+1)}{1-x^2}y = 0.$$

The singular points are $x = \pm 1$. For $x_0 = 0$ (ordinary point), we use power series.

Solution. Substituting $y = \sum_{k=0}^{\infty} c_k x^k$ into Legendre's equation yields the recurrence

$$c_{k+2} = \frac{k(k+1) - n(n+1)}{(k+1)(k+2)}c_k, \quad k \geq 0.$$

Two linearly independent solutions are obtained from $c_0 = 1, c_1 = 0$ (even series) and $c_0 = 0, c_1 = 1$ (odd series).

Legendre Polynomials. When n is a non-negative integer, one of the series terminates, producing a polynomial of degree n. The Legendre polynomial $P_n(x)$, normalized so that $P_n(1) = 1$, is given by

$$P_n(x) = \frac{1}{2^n}\sum_{m=0}^{\lfloor n/2 \rfloor}(-1)^m\binom{n}{m}\binom{2n-2m}{n}x^{n-2m}.$$

The first few Legendre polynomials are:

$$P_0(x) = 1, \quad P_1(x) = x, \quad P_2(x) = \tfrac{1}{2}(3x^2-1),$$
$$P_3(x) = \tfrac{1}{2}(5x^3-3x), \quad P_4(x) = \tfrac{1}{8}(35x^4 - 30x^2 + 3).$$

Rodrigues' Formula. $P_n(x) = \dfrac{1}{2^n n!}\dfrac{d^n}{dx^n}(x^2-1)^n$.

Orthogonality. $\displaystyle\int_{-1}^{1} P_m(x)P_n(x)\,dx = \frac{2}{2n+1}\delta_{mn}$, where δ_{mn} is the Kronecker delta.

Recurrence. $(n+1)P_{n+1}(x) = (2n+1)xP_n(x) - nP_{n-1}(x)$.

Generating Function. $\displaystyle\frac{1}{\sqrt{1-2xt+t^2}} = \sum_{n=0}^{\infty} P_n(x)t^n, \quad |t| < 1.$

37 Bessel Differential Equation

The differential equation of the form

$$x^2y'' + xy' + (x^2 - \alpha^2)y = 0,$$

where $\alpha \geq 0$ is called the Bessel's equation of order α. Here $x_0 = 0$ is a regular singular point. The indicial equation is $r^2 - \alpha^2 = 0$, giving $r = \pm\alpha$.

Bessel Function of the First Kind. For $r = \alpha$, the Frobenius method yields

$$J_\alpha(x) = \sum_{m=0}^{\infty}\frac{(-1)^m}{m!\,\Gamma(m+\alpha+1)}\left(\frac{x}{2}\right)^{2m+\alpha}.$$

The first few are:

$$J_0(x) = 1 - \frac{x^2}{4} + \frac{x^4}{64} - \dots, \quad J_1(x) = \frac{x}{2} - \frac{x^3}{16} + \frac{x^5}{384} - \dots$$

Bessel Function of the Second Kind. When $\alpha \notin \mathbb{Z}$, a second linearly independent solution is $J_{-\alpha}(x)$. When $\alpha = n \in \mathbb{Z}^+ \cup \{0\}$, $J_{-n} = (-1)^n J_n$ (linearly dependent), and the second solution is the Bessel function of the second kind:

$$Y_n(x) = \lim_{\alpha\to n}\frac{J_\alpha(x)\cos(\alpha\pi) - J_{-\alpha}(x)}{\sin(\alpha\pi)}.$$

The general solution of Bessel's equation of integer order n is

$$y = C_1 J_n(x) + C_2 Y_n(x).$$

Properties.

1. Recurrence: $J_{\alpha-1}(x) + J_{\alpha+1}(x) = \dfrac{2\alpha}{x}J_\alpha(x)$.
2. Derivative: $\dfrac{d}{dx}[x^\alpha J_\alpha(x)] = x^\alpha J_{\alpha-1}(x)$.
3. $\dfrac{d}{dx}[x^{-\alpha}J_\alpha(x)] = -x^{-\alpha}J_{\alpha+1}(x)$.
4. Orthogonality: $\displaystyle\int_0^1 xJ_\alpha(\lambda_m x)J_\alpha(\lambda_n x)\,dx = \frac{\delta_{mn}}{2}[J_{\alpha+1}(\lambda_n)]^2$, where λ_n are zeros of J_α.

Parametric Bessel Equation. The equation $x^2y'' + xy' + (k^2x^2 - \alpha^2)y = 0$ has the general solution $y = C_1 J_\alpha(kx) + C_2 Y_\alpha(kx)$.

38 Chebyshev Differential Equation

The equation of the form

$$(1-x^2)y'' - xy' + n^2 y = 0$$

is called Chebyshev's equation. The singular points are $x = \pm 1$, and $x_0 = 0$ is an ordinary point.

Solution. Substituting $y = \sum_{k=0}^{\infty} c_k x^k$ gives the recurrence

$$c_{k+2} = \frac{k^2 - n^2}{(k+1)(k+2)} c_k.$$

When n is a non-negative integer, one series terminates, producing a polynomial.

Chebyshev Polynomials of the First Kind. The Chebyshev polynomial $T_n(x)$ satisfies $T_n(\cos\theta) = \cos(n\theta)$. Explicitly:

$$T_n(x) = \frac{n}{2} \sum_{m=0}^{\lfloor n/2 \rfloor} \frac{(-1)^m (n-m-1)!}{m!(n-2m)!} (2x)^{n-2m}, \quad n \geq 1.$$

The first few are:

$$T_0(x) = 1,\ T_1(x) = x,\ T_2(x) = 2x^2 - 1,$$
$$T_3(x) = 4x^3 - 3x,\ T_4(x) = 8x^4 - 8x^2 + 1.$$

Recurrence. $T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x)$.

Orthogonality. $\displaystyle\int_{-1}^{1} \frac{T_m(x)T_n(x)}{\sqrt{1-x^2}}\, dx = \begin{cases} \pi, & m = n = 0 \\ \pi/2, & m = n \neq 0 \\ 0, & m \neq n \end{cases}$

Chebyshev Polynomials of the Second Kind. $U_n(x)$ satisfies $U_n(\cos\theta) = \frac{\sin((n+1)\theta)}{\sin\theta}$. Recurrence: $U_{n+1}(x) = 2xU_n(x) - U_{n-1}(x)$, with $U_0 = 1$, $U_1 = 2x$.

39 Airy Differential Equation

An equation of the form

$$y'' - xy = 0$$

is called the Airy differential equation. Here $x_0 = 0$ is an ordinary point.

Solution. Substituting $y = \sum_{n=0}^{\infty} c_n x^n$ gives (as derived in the Power Series section):

$$c_2 = 0, \quad c_{n+2} = \frac{c_{n-1}}{(n+2)(n+1)}, \quad n \geq 1.$$

Two linearly independent solutions are the Airy functions $\mathrm{Ai}(x)$ and $\mathrm{Bi}(x)$:

$$\mathrm{Ai}(x) = c_1 f(x) - c_2 g(x), \quad \mathrm{Bi}(x) = \sqrt{3}\left(c_1 f(x) + c_2 g(x)\right),$$

where

$$f(x) = 1 + \sum_{k=1}^{\infty} \frac{x^{3k}}{(2\cdot 3)(5\cdot 6)\cdots((3k-1)\cdot 3k)},$$
$$g(x) = x + \sum_{k=1}^{\infty} \frac{x^{3k+1}}{(3\cdot 4)(6\cdot 7)\cdots(3k\cdot(3k+1))},$$

and $c_1 = \mathrm{Ai}(0) = \dfrac{1}{3^{2/3}\Gamma(2/3)}$, $c_2 = -\mathrm{Ai}'(0) = \dfrac{1}{3^{1/3}\Gamma(1/3)}$.

Integral Representation.

$$\mathrm{Ai}(x) = \frac{1}{\pi}\int_0^{\infty} \cos\left(\frac{t^3}{3} + xt\right) dt.$$

Asymptotic Behavior. As $x \to +\infty$:

$$\mathrm{Ai}(x) \sim \frac{e^{-\frac{2}{3}x^{3/2}}}{2\sqrt{\pi}\, x^{1/4}}, \quad \mathrm{Bi}(x) \sim \frac{e^{\frac{2}{3}x^{3/2}}}{\sqrt{\pi}\, x^{1/4}}.$$

40 Hypergeometric Differential Equation

The equation of the form

$$x(1-x)y'' + [\gamma - (\alpha+\beta+1)x]y' - \alpha\beta\gamma = 0,$$

where $\alpha, \beta, \gamma \in \mathbb{C}$, is called the hypergeometric equation.

Substitute

$$y = \sum_{n=1}^{\infty} a_n x^n$$

in, and then solve for a_n, we obtain

$$a_n = \frac{(n+\alpha-1)(n+\beta-1)}{n(n+\gamma-1)} a_{n-1}$$

Solving recursively for a_n,

$$a_n = \frac{\alpha(\alpha+1)\ldots(\alpha+n-1)\beta(\beta+1)\ldots(\beta+n-1)}{n!\gamma(\gamma+1)\ldots(\gamma+n-1)}$$

The rising factorial, $(t)_n$ is defined as

$$(t)_n = \begin{cases} t(t+1)\ldots(t+n-1), & \text{if } n \geq 1 \\ 1, & \text{if } n = 0 \end{cases}$$

Apply this definition in, we have

$$a_n = \frac{(\alpha)_n(\beta)_n}{n!(\gamma)_n}, \quad n \geq 1$$

41 Laguerre Differential Equation

The equation of the form

$$xy'' + (1-x)y' + ny = 0$$

where $n \geq 0$ is a parameter, is called Laguerre's equation. Here $x_0 = 0$ is a regular singular point.

Solution. Substituting a power series yields the recurrence

$$c_{k+1} = \frac{k-n}{(k+1)^2} c_k.$$

When n is a non-negative integer, the series terminates, producing the Laguerre polynomial $L_n(x)$.

Laguerre Polynomials. The Laguerre polynomial of degree n is

$$L_n(x) = \sum_{k=0}^{n} \frac{(-1)^k n!}{(k!)^2(n-k)!} x^k.$$

The first few are:

$$L_0(x) = 1,\ L_1(x) = 1 - x,\ L_2(x) = 1 - 2x + \tfrac{x^2}{2},$$
$$L_3(x) = 1 - 3x + \tfrac{3x^2}{2} - \tfrac{x^3}{6}.$$

Rodrigues' Formula. $L_n(x) = \dfrac{e^x}{n!}\dfrac{d^n}{dx^n}(x^n e^{-x})$.

Orthogonality. $\displaystyle\int_0^{\infty} e^{-x} L_m(x) L_n(x)\, dx = \delta_{mn}$.

Recurrence. $(n+1)L_{n+1}(x) = (2n+1-x)L_n(x) - nL_{n-1}(x)$.

Associated Laguerre Polynomials. The associated Laguerre polynomial $L_n^{(\alpha)}(x)$ satisfies $xy'' + (1+\alpha-x)y' + ny = 0$, and is defined by

$$L_n^{(\alpha)}(x) = \sum_{k=0}^{n} \frac{(-1)^k \binom{n+\alpha}{n-k}}{k!} x^k.$$

Orthogonality: $\displaystyle\int_0^{\infty} x^{\alpha} e^{-x} L_m^{(\alpha)}(x) L_n^{(\alpha)}(x)\, dx = \frac{\Gamma(n+\alpha+1)}{n!}\delta_{mn}$.

42 Hermite Differential Equation

The equation of the form

$$y'' - 2xy' + 2ny = 0$$

where $n \geq 0$ is a parameter, is called Hermite's equation. Here $x_0 = 0$ is an ordinary point.

Solution. Substituting $y = \sum_{k=0}^{\infty} c_k x^k$ yields the recurrence

$$c_{k+2} = \frac{2(k-n)}{(k+1)(k+2)} c_k.$$

When n is a non-negative integer, one series terminates, producing the Hermite polynomial $H_n(x)$.

Hermite Polynomials. The Hermite polynomial of degree n is

$$H_n(x) = n! \sum_{m=0}^{\lfloor n/2 \rfloor} \frac{(-1)^m}{m!(n-2m)!}(2x)^{n-2m}.$$

The first few are:

$$H_0(x) = 1,\ H_1(x) = 2x,\ H_2(x) = 4x^2 - 2,$$
$$H_3(x) = 8x^3 - 12x,\ H_4(x) = 16x^4 - 48x^2 + 12.$$

Rodrigues' Formula. $H_n(x) = (-1)^n e^{x^2} \dfrac{d^n}{dx^n} e^{-x^2}$.

Orthogonality. $\int_{-\infty}^{\infty} e^{-x^2} H_m(x)H_n(x)\,dx = \sqrt{\pi}\,2^n n!\,\delta_{mn}$.

Recurrence. $H_{n+1}(x) = 2xH_n(x) - 2nH_{n-1}(x)$.

Generating Function. $e^{2xt-t^2} = \sum_{n=0}^{\infty} H_n(x)\frac{t^n}{n!}$.

Application. The Hermite polynomials appear in the solution of the quantum harmonic oscillator. The normalized wave functions are $\psi_n(x) = \frac{1}{\sqrt{2^n n!\sqrt{\pi}}} e^{-x^2/2} H_n(x)$.

EIGENVALUE PROBLEM

43 Sturm-Liouville Problem

Standard Form. A Sturm-Liouville problem is a BVP of the form

$$\frac{d}{dx}\left[p(x)\frac{dy}{dx}\right] - q(x)y + \lambda r(x)y = 0, \quad a < x < b,$$

together with boundary conditions

$$\alpha_1 y(a) + \alpha_2 y'(a) = 0, \quad \beta_1 y(b) + \beta_2 y'(b) = 0,$$

where $p(x) > 0$, $r(x) > 0$, $q(x) \geq 0$ on $[a, b]$, and λ is a parameter (the eigenvalue).

Properties.

1. All eigenvalues λ_n are real.
2. The eigenvalues form an increasing sequence $\lambda_1 < \lambda_2 < \lambda_3 < \dots$ with $\lambda_n \to \infty$ as $n \to \infty$.
3. For each λ_n, there is a corresponding eigenfunction $\phi_n(x)$ (unique up to a constant multiple).
4. Eigenfunctions corresponding to distinct eigenvalues are orthogonal with respect to the weight function $r(x)$:
$$\int_a^b r(x)\phi_m(x)\phi_n(x)\,dx = 0, \quad m \neq n.$$
5. The eigenfunctions $\{\phi_n\}$ form a complete set: any piecewise smooth function $f(x)$ on $[a, b]$ can be expanded as
$$f(x) = \sum_{n=1}^{\infty} c_n\phi_n(x), \quad \text{where } c_n = \frac{\int_a^b r(x)f(x)\phi_n(x)\,dx}{\int_a^b r(x)[\phi_n(x)]^2\,dx}.$$

Example. Find the eigenvalues and eigenfunctions of

$$y'' + \lambda y = 0, \quad y(0) = 0,\ y(L) = 0.$$

Solution. This is a Sturm-Liouville problem with $p = 1$, $q = 0$, $r = 1$.

- $\lambda < 0$: $y = C_1 e^{\sqrt{-\lambda}x} + C_2 e^{-\sqrt{-\lambda}x}$. Applying BCs gives $y \equiv 0$ (trivial).
- $\lambda = 0$: $y = Ax + B$. Applying BCs gives $y \equiv 0$.
- $\lambda > 0$: $y = A\cos(\sqrt{\lambda}x) + B\sin(\sqrt{\lambda}x)$. $y(0) = 0 \Rightarrow A = 0$. $y(L) = 0 \Rightarrow B\sin(\sqrt{\lambda}L) = 0$. For nontrivial y, we need $\sqrt{\lambda}L = n\pi$, thus
$$\lambda_n = \frac{n^2\pi^2}{L^2}, \quad \phi_n(x) = \sin\frac{n\pi x}{L}, \quad n = 1, 2, 3, \dots$$

Rayleigh Quotient. For the Sturm-Liouville problem, the eigenvalues satisfy

$$\lambda = \frac{-p(x)y\,y'\Big|_a^b + \int_a^b \left[p(x)(y')^2 + q(x)y^2\right]dx}{\int_a^b r(x)y^2\,dx}.$$

44 Green's Functions

Motivation. Green's functions provide a systematic method for solving non-homogeneous boundary value problems. The idea is to express the solution of $Ly = F(x)$ with BCs as an integral involving a kernel $G(x, t)$ — the Green's function — which encodes both the differential operator and the boundary conditions.

Setup. Consider the BVP

$$Ly = y'' + p(x)y' + q(x)y = F(x), \quad a < x < b,$$

with homogeneous boundary conditions $B_1[y] = 0$, $B_2[y] = 0$. Assume that the associated homogeneous problem $Ly = 0$ with the same BCs has only the trivial solution $y \equiv 0$.

Definition. The Green's function $G(x, t)$ for the operator L with boundary conditions B_1, B_2 is the function satisfying:

1. $G(x, t)$ satisfies $L_xG = 0$ for $x \neq t$ (homogeneous DE in x).
2. $G(x, t)$ satisfies the boundary conditions in x: $B_1[G] = 0$, $B_2[G] = 0$.
3. $G(x, t)$ is continuous at $x = t$: $G(t^+, t) = G(t^-, t)$.
4. G_x has a jump discontinuity at $x = t$: $G_x(t^+, t) - G_x(t^-, t) = 1$.

Solution Formula. Once $G(x, t)$ is found, the solution to the BVP is

$$y(x) = \int_a^b G(x, t)F(t)\,dt.$$

Construction. Let $y_1(x)$ satisfy $Ly = 0$ with $B_1[y_1] = 0$, and $y_2(x)$ satisfy $Ly = 0$ with $B_2[y_2] = 0$. Then,

$$G(x, t) = \begin{cases} \dfrac{y_1(x)y_2(t)}{W(t)}, & a \leq x \leq t \\ \dfrac{y_1(t)y_2(x)}{W(t)}, & t \leq x \leq b \end{cases}$$

where $W(t) = y_1(t)y_2'(t) - y_1'(t)y_2(t)$ is the Wronskian.

Properties.

1. Symmetry: $G(x, t) = G(t, x)$ (for self-adjoint operators).
2. $G(x, t)$ exists and is unique if and only if the homogeneous problem has only the trivial solution.

Example. Find the Green's function for $y'' = F(x)$, $y(0) = 0$, $y(1) = 0$.
Solution. The homogeneous solutions are $y_1(x) = x$ (satisfies $y(0) = 0$) and $y_2(x) = 1 - x$ (satisfies $y(1) = 0$). The Wronskian is

$$W(t) = x(-(1)) - 1 \cdot (1 - x) = -x - 1 + x = -1.$$

Therefore,

$$G(x, t) = \begin{cases} -x(1 - t), & 0 \leq x \leq t \\ -t(1 - x), & t \leq x \leq 1 \end{cases}$$

and the solution to $y'' = F(x)$, $y(0) = y(1) = 0$ is

$$y(x) = \int_0^1 G(x, t)F(t)\,dt = -\int_0^x t(1 - x)F(t)\,dt - \int_x^1 x(1 - t)F(t)\,dt.$$

Connection to Sturm-Liouville Theory. For the Sturm-Liouville operator $Ly = \frac{d}{dx}\left[p(x)y'\right] - q(x)y$, the Green's function can be expanded in terms of the eigenfunctions $\{\phi_n\}$:

$$G(x, t) = \sum_{n=1}^{\infty} \frac{\phi_n(x)\phi_n(t)}{\lambda_n\|\phi_n\|^2},$$

where λ_n are the eigenvalues and $\|\phi_n\|^2 = \int_a^b r(x)[\phi_n(x)]^2\,dx$.

Green's Function for IVPs. For the IVP $Ly = F(x)$, $y(a) = 0$, $y'(a) = 0$, the Green's function is the causal (one-sided) kernel:

$$G(x, t) = \begin{cases} \dfrac{y_1(t)y_2(x) - y_1(x)y_2(t)}{W(t)}, & a \leq t \leq x \\ 0, & x < t \end{cases}$$

and the solution is $y(x) = \int_a^x G(x, t)F(t)\,dt$.

FOURIER SERIES

45 Definitions and Formulas

Fourier Series. Let $f(x)$ be a piecewise smooth, periodic function with period $2L$. The Fourier series of f is

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi x}{L} + b_n\sin\frac{n\pi x}{L}\right),$$

where the Fourier coefficients are

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(x)\,dx,$$
$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx, \quad n = 1, 2, 3, \dots$$
$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx, \quad n = 1, 2, 3, \dots$$

Theorem 14 (**Fourier Convergence**)**.** If $f(x)$ is piecewise smooth on $[-L, L]$, then the Fourier series converges to

$$\frac{f(x^+) + f(x^-)}{2}$$

at every point $x \in (-L, L)$. At points of continuity, it converges to $f(x)$.

Even and Odd Functions.

- If f is even ($f(-x) = f(x)$), then $b_n = 0$ for all n. The series is a **Fourier cosine series**:
$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L}, \quad a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx.$$
- If f is odd ($f(-x) = -f(x)$), then $a_n = 0$ for all n. The series is a **Fourier sine series**:
$$f(x) \sim \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}, \quad b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx.$$

46 Half-Range Expansions

If $f(x)$ is defined only on $[0, L]$, we can extend it to $[-L, L]$ as either even or odd:

- **Cosine series (even extension):** Extend f so that $f(-x) = f(x)$. Use the cosine series formula.
- **Sine series (odd extension):** Extend f so that $f(-x) = -f(x)$. Use the sine series formula.

Example. Find the Fourier sine series of $f(x) = x$ on $[0, L]$.
Solution.
$$\begin{aligned} b_n &= \frac{2}{L}\int_0^L x\sin\frac{n\pi x}{L}\,dx = \frac{2}{L}\left[-\frac{Lx}{n\pi}\cos\frac{n\pi x}{L} + \frac{L^2}{n^2\pi^2}\sin\frac{n\pi x}{L}\right]_0^L \\ &= \frac{-2L}{n\pi}\cos(n\pi) = \frac{2L(-1)^{n+1}}{n\pi} \end{aligned}$$
Therefore,
$$x = \frac{2L}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin\frac{n\pi x}{L}, \quad 0 < x < L.$$

47 Parseval's Theorem

Theorem 15 (**Parseval's Identity**). If $f(x)$ has Fourier series on $[-L, L]$, then
$$\frac{1}{L}\int_{-L}^{L}[f(x)]^2\,dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty}(a_n^2 + b_n^2).$$

48 Complex Fourier Series

The complex form of the Fourier series is
$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{in\pi x/L}, \quad \text{where} \quad c_n = \frac{1}{2L}\int_{-L}^{L} f(x)e^{-in\pi x/L}\,dx.$$
The relationship between real and complex coefficients is $c_0 = a_0/2$, and for $n \geq 1$: $c_n = (a_n - ib_n)/2$, $c_{-n} = (a_n + ib_n)/2$.

BOUNDARY VALUE PROBLEM

49 Introduction

Boundary Value Problem. A BVP consists of a DE together with conditions imposed at two or more points. For a second-order DE $y'' + p(x)y' + q(x)y = F(x)$ on $[a, b]$:
$$\alpha_1 y(a) + \alpha_2 y'(a) = \gamma_1, \quad \beta_1 y(b) + \beta_2 y'(b) = \gamma_2.$$
Unlike IVPs, BVPs may have zero, one, or infinitely many solutions.

Types of Boundary Conditions.

1. **Dirichlet:** $y(a) = \alpha$, $y(b) = \beta$ (value prescribed).
2. **Neumann:** $y'(a) = \alpha$, $y'(b) = \beta$ (derivative prescribed).
3. **Robin (Mixed):** $\alpha_1 y(a) + \alpha_2 y'(a) = \gamma$ (combination).
4. **Periodic:** $y(a) = y(b)$, $y'(a) = y'(b)$.

50 Separation of Variables

Strategy. For a PDE such as $u_t = ku_{xx}$, assume $u(x,t) = X(x)T(t)$. Substitute into the PDE and separate variables to obtain two ODEs connected by a separation constant $-\lambda$. The boundary conditions determine the eigenvalue problem for X, and the initial conditions determine T.

51 The Heat Equation

The one-dimensional heat equation is
$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}, \quad 0 < x < L,\ t > 0,$$
with boundary conditions $u(0,t) = 0$, $u(L,t) = 0$, and initial condition $u(x,0) = f(x)$.

Solution by Separation of Variables. Let $u(x,t) = X(x)T(t)$. Substituting:
$$\frac{T'}{kT} = \frac{X''}{X} = -\lambda \quad \text{(separation constant)}.$$
This gives the eigenvalue problem $X'' + \lambda X = 0$, $X(0) = 0$, $X(L) = 0$, and $T' + k\lambda T = 0$. The eigenvalues and eigenfunctions are
$$\lambda_n = \frac{n^2\pi^2}{L^2}, \quad X_n(x) = \sin\frac{n\pi x}{L}.$$
The time part is $T_n(t) = e^{-kn^2\pi^2 t/L^2}$. By superposition:
$$u(x,t) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L}\, e^{-kn^2\pi^2 t/L^2},$$
where $B_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx$.

Insulated Ends. For $u_x(0,t) = u_x(L,t) = 0$ (Neumann BCs):
$$u(x,t) = \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n \cos\frac{n\pi x}{L}\, e^{-kn^2\pi^2 t/L^2},$$
where $A_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx$.

52 The Wave Equation

The one-dimensional wave equation is
$$\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2}, \quad 0 < x < L,\ t > 0,$$
with BCs $u(0,t) = u(L,t) = 0$, and ICs $u(x,0) = f(x)$, $u_t(x,0) = g(x)$.

Solution. Using separation of variables, the solution is
$$u(x,t) = \sum_{n=1}^{\infty}\left(A_n\cos\frac{cn\pi t}{L} + B_n\sin\frac{cn\pi t}{L}\right)\sin\frac{n\pi x}{L},$$
where
$$A_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx, \quad B_n = \frac{2}{cn\pi}\int_0^L g(x)\sin\frac{n\pi x}{L}\,dx.$$

D'Alembert's Solution. For the wave equation on $(-\infty, \infty)$ with $u(x,0) = f(x)$ and $u_t(x,0) = g(x)$:
$$u(x,t) = \frac{1}{2}[f(x+ct) + f(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\,ds.$$

53 Laplace's Equation

Laplace's equation in two dimensions is
$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0,$$
also written $\nabla^2 u = 0$. Solutions are called harmonic functions.

Dirichlet Problem on a Rectangle. For the rectangle $0 < x < a$, $0 < y < b$ with $u(x,0) = 0$, $u(x,b) = 0$, $u(0,y) = 0$, and $u(a,y) = f(y)$:
$$u(x,y) = \sum_{n=1}^{\infty} B_n \frac{\sinh\frac{n\pi x}{b}}{\sinh\frac{n\pi a}{b}}\sin\frac{n\pi y}{b},$$
where $B_n = \frac{2}{b}\int_0^b f(y)\sin\frac{n\pi y}{b}\,dy$.

Dirichlet Problem on a Disk. For the disk $r < R$ in polar coordinates with $u(R,\theta) = f(\theta)$:
$$u(r,\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(\frac{r}{R}\right)^n (a_n\cos n\theta + b_n\sin n\theta),$$
where a_n and b_n are the Fourier coefficients of $f(\theta)$ with $L = \pi$.

Poisson's Integral Formula. For the disk $r < R$:
$$u(r,\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{R^2 - r^2}{R^2 - 2Rr\cos(\theta - \phi) + r^2} f(\phi)\,d\phi.$$